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Fully Modified HP Filter

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Abstract

Business cycle estimation is core of macroeconomics research. Hodrick-Prescott (1997) filter, (or HP filter), is the most popular tool to extract cycle from a macroeconomic time series. There are certain issues with HP filter including fixed value of λ across the series/countries and end points bias (EPB). Modified HP filter (MHP) of McDermott (1997) attempted to address the first issue. Bloechl (2014) introduced a loss function minimization approach to address the EPB issue but keeping lambda fixed (as in HP filter). In this study we marry the endogenous lambda approach of McDermott (1997) with loss function minimization approach of Bloechl (2014) to analyze EPB in HP filter, while intuitively changing the weighting scheme used in the latter. We contribute by suggesting an endogenous weighting scheme along with endogenous smoothing parameter to resolve EPB issue of HP filter. We call this fully modified HP (FMHP) filter. Our FMHP filter outperforms a variety of conventional filters in a power comparison (simulation) study as well as in observed real data (univariate and multivariate) analytics for a large set of countries.

Key Words: Business Cycle, Time Series, Fully Modified HP Filter, End Point Bias in HP Filter, Simulation, Cross Country Study.
JEL Codes: E32, C18

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Non-technical Summary

Hodrick-Prescott (1997) filter is the most popular tool to extract trend and cycle components from a time series. There are certain issues with HP filter including i) fixing the value of smoothing parameter (lambda) across the series/countries and ii) end points bias (EPB) in the extracted cycle component. Modified HP filter (MHP) of McDermott (1997) attempted to address the first issue and suggested estimating lambda endogenously. Issue of EPB is still unresolved despite some suggestions like extrapolating the subject time series. EPB issue originates from asymmetrical weights for significant number of terminal observations in HP filtering process. Bloechl (2014) introduced a loss function minimization approach to address the EPB issue but keeping lambda fixed. He suggested varying ‘number’ of end observations upon which different weights are applied.

In this study we marry the endogenous lambda approach of McDermott (1997) with loss function minimization approach of Bloechl (2014) to analyze EPB in HP filter, while suggesting some innovative changes in weighting scheme used in the later. We contribute by suggesting an endogenous weighting scheme, which is different from Bloechl (2014), along with endogenous smoothing parameter and reduce EPB in HP filter. We call this *fully modified HP* (FMHP) *filter*.

Our FMHP filter outperforms the conventional filters (like HP, CF filters and wavelet analysis with extrapolated data) in a power comparison study. We also apply FMHP filtering upon 3 core macroeconomic time series - namely real income, consumption and investment - for annual (quarterly) data of a large number of countries and find that our FMHP filter lowers the estimated loss compared to other filters drastically. FMHP filter performs better than HP filter in moments’ analytics of detrended real income, consumption and investment series.

1. Introduction

Business cycle estimation is considered as the core of macroeconomics research. Hodrick-Prescott (1997) filter, (or HP filter), is the most popular and widely used tool to extract cyclical component from a macroeconomic time series. However, there are certain issues with HP filter which are well known in the literature [see McDermott (1997), Kaiser and Maravall (1999), Ekinci et al. (2013), Choudhary et al (2014), and Bloechl, (2014)] including (i) fixing the value of smoothing parameter, i.e. λ (lambda), across the series/countries and (ii) end points bias (EPB) in the extracted cyclical component.

Modified HP filter (MHP) of McDermott (1997) attempted to address the first issue and suggested estimating smoothing parameter (lambda) endogenously. Choudhary et al (2014) assessed MHP filter and found that it performs better than a list of competing filtering approaches, including HP filter, for detrending time series' and that the filtering method matters in actual/observed macroeconomics time series univariate as well as multivariate data analytics. Issue of EPB is still unresolved[1].

EPB contaminates the estimated trend with the cyclical component and thus underestimate the cyclical component during both the recovery as well as recession. It also results in downward biased standard error of the estimated cyclical component. A downward biased standard error of the estimated cyclical component may give impression of a stable economy, and the underestimated cyclical component during booming/receding economy may delay the necessary stabilization measures by economic managers.

Essentially, root of EPB issue lies in asymmetrical weights for significant number of terminal observations in HP filtering process (Figure 1 of Appendix B). Bloechl (2014) introduced a loss function minimization approach to address the EPB issue but keeping lambda fixed as in HP filter. He suggested varying number of end observations upon which different weights are applied. Bloechl (2014) found estimated loss, with his suggested weighting scheme, to be lower than that for HP filter approach.

In this study we marry the endogenous lambda approach of McDermott (1997) [studied in Choudhary et al (2014)] with the loss function minimization approach of Bloechl (2014) to address EPB in HP filter, along with suggesting some innovative changes in weighting scheme used in the later. We contribute by suggesting an endogenous weighting scheme, which is intuitively better than used in Bloechl (2014), along with endogenous smoothing parameter (as in McDermott, 1997) to resolve EPB issue of HP filter. We call this fully modified HP (FMHP) filter.

Our FMHP filter outperforms completely the conventional filters like Hodrick and Prescott (1997) [or HP], wavelet analysis based filtering with extrapolated data (see Iqbal and Hanif, 2017) [or WAN-WED], and Christiano and Fitzgerald (2003) [or CF] in a power comparison (simulation) study (Table 1, Appendix A) employing data generating models used in Choudhary et al (2014). End point performance of our FMHP filter is specifically evaluated (compared to HP, WAN-WED and CF filters) and found best (Table 2, Appendix A).

We also apply FMHP filtering upon three core macroeconomic time series - namely real income (Y), consumption (C) and investment (I) - for annual (quarterly) data of 70 (33) countries. In observed data

[1] Some studies like (Kaiser and Maravall, 1999) suggested extrapolating the subject time series before applying the HP filter as a temporary solution. Extrapolated observations in themselves would be biased to the extent the goodness of fit will not be achieved in modeling the underlying series of interest.

application we find that our FMHP filter lowers the estimated loss drastically compared to those obtained using Bloechl (2014) as well as HP filter (Tables 3a and 3b of Appendix A and Figures 3 and 3a of Appendix B). Research based on observed data of real income, investment and consumption also shows that the autoregressive properties and multivariate analytics of cyclical components depend upon filtering technique applied and that our FMHP filter performs better (Table 4, Table 5, Appendix A).

Remainder of this study is organized as follows: In Section 2, we explain what is inside HP filtering process. In fact we set locale here to show where the issues in the HP filter are which we explain in the next section, focusing on end point bias. Some of the proposal to address EPB are discussed and critically evaluated in Section 4. In Section 5 we propose our FMHP filter which lowers the EPB drastically as shown in the later part of this section in simulation and empirical applications. Section 6 is for conclusion.

2. What is in the HP Filter?

Hodrick and Prescott (1997) filter decomposes a time series x_t in to trend (g_t) and cyclical (c_t) components so that

$$x_t = g_t + c_t \quad (1)$$

The trend component g_t is estimated by solving the minimization problem

$$min[\sum_{t=1}^{T}(x_t - g_t)^2 + \lambda \sum_{t=2}^{T-1}[(g_{t+1} - g_t) - (g_t - g_{t-1})]^2] \quad (2)$$

In this optimization problem, there is a trade-off between the 'goodness of fit' and the 'degree of smoothness' that depends on the value of λ. The solution to the minimization problem in (2) for g_t is

$$\hat{g}_t = [I + \lambda A]^{-1} x_t = B x_t \qquad \text{(details are in Appendix C)} \quad (3)$$

Where $A = K'K$ where $K = \{k_{ij}\}$ is a (T-2) ×T matrix with elements as given below

$$k_{ij} = \begin{cases} 1 & if\ j = i\ or\ j = i+2, \\ -2 & if\ \ j = i+1, \\ 0 & otherwise \end{cases}$$

Hodrick and Prescott (1997) applied this procedure on quarterly seasonally adjusted GDP data of USA by fixing[2] the value of λ at 1600 to estimate cyclical (transitory) and trend (permanent) component.

3. Issues with HP Filter and its Use

There are certain issues with HP filter as have been highlighted in the literature from time to time (McDermott (1997), Kaiser and Maravall (1999), Mise et al (2005), Ekinci et al. (2013), Choudhary et al (2014), and Bloechl (2014)). We discuss two of the issues with HP filter in the following subsections.

[2] Based upon their view that a 5 percent cyclical component is moderately large, as is one-eighth of 1 percent change in the (real GDP) growth rate in a quarter. This led them to select smoothing parameter for quarterly data as 1600.

3.1. Fixed value of Smoothing Parameter

One of these issues pertains to the way HP filter has been put in practice by empirical researchers. Unfortunately, it has become a convention in empirical economic research to use $\lambda = 1600$ (100) for smoothing quarterly (annual) time series across the economies / series. We are of the view that λ must be endogenous (to the actually observed data) as has been argued in McDermott (1997) and validated in Choudhary et al. (2014). Thus, it could be different even for USA GDP in case of time period different from Hodrick and Prescott (1997) analysed. This is because the value of the smoothing parameter of a given series relies on the underlying behavior of economic agents from where its dynamical properties originate, and this fact cannot be ignored.

Modified HP filter of McDermott (1997) relaxes the assumption of fixed λ as explained in Choudhary et al (2014). The idea of McDermott (1997) is to estimate the trend component as in equation (3) by excluding a single data point at a time and selecting a value of λ which gives best fit of the left out data point. This 'leave out method' is applied to all the data points one by one. The optimal value of λ can be obtained by minimizing the following gross cross validation (GCV) function with respect to λ.

$$\mathrm{GCV}(\lambda) = T^{-1}\left(1 + \frac{2T}{\lambda}\right)\sum_{k=1}^{T}(x_k - g_{t,k}(\lambda))^2 \quad (4)$$

In this way smoothing parameter becomes endogenous (to observed data dynamics).

3.2. End Point Bias

The other issue pertains not to the way HP filter is being used in practice but lies in the construction of HP filter. In order to carry out a trend-cycle decomposition of a time series at a given date, HP filtering requires information about the behavior of the series at prior as well as at later dates. Absence of end and start observations (as is evident from equation 2) poses difficulties at the start and end of the sample resulting in changes in terminal points weights and thus causing substantial distortions in cyclical component at both ends. This is what has been termed as end-point bias in the literature (Baxter and King, 1999 [or BK], Mise et al (2005) and Auria et.al, 2010). We explain this problem in following with the help of a figure (Figure 1, of Appendix B).

HP filter is a symmetric filter in the sense that the estimator $\hat{g}_t$ (equation 3) is the weighted sum of both lags and leads of x_t. Due to the missing values at both ends, the whole weighted matrix B in equation (3) is distorted with highest effect on boundaries and lowest at the middle of the data set (Baxter and King, 1999). This can be seen from Figure 1 (Appendix B), where we plot the 'weighting vectors' corresponding to $\hat{g}_t$ for a HP-filter with $\lambda = 100$ applied to a series with 50 data points. We can see that as we go to the middle of data set we have symmetric weighting vector while towards the both ends the filter weights become more and more asymmetric. It can be observed that the highest weight at the margins is disproportionately large compared to those at the middle (of the time series). Hence the estimation at the end points is effected by disproportionately large weights at terminal points. This behavior of HP filter weighting scheme causes biasness at (up to 20 observations on[3]) both ends in the extracted cyclical component.

[3] According to our own simulation exercise.

To further understand the distortion at the terminal points of the estimated trend component of time series using HP filter, we can consider HP filter in frequency domain (as in Bloechl (2014) for example).

Going back to equation (3) we see the filter weights are contained in the matrix $[\mathrm{I} + \lambda \mathrm{A}]^{-1} = \mathrm{B}_{\mathrm{T}\times\mathrm{T}}$, where B is symmetric matrix and its t^{th} row contains the weights for the estimation $\hat{g}_t$

$$\hat{g}_t = \sum_{j=1}^{T} b_{tj} x_j \qquad (5)$$

Where b_{tj} is the j^{th} element of the t^{th} row of B. These weights have symmetric structure in the middle[4] and change near the boundaries[5].

In the frequency domain a time series is interpreted as an overlap of oscillations with different frequencies (Harvey and Jaeger (1993) and Hamilton (1994)) where the trend as a long run behavior (of a series) is supposed to consist of those oscillations with high periodicity. In the estimation of trend component, HP filter extracts oscillations with high periodicities and eliminates oscillations with lower periodicities. This behavior can be explained by a gain function. By using filter weights in matrix B for a given value of lambda, (following Mills, 2003) the gain for estimation $\hat{g}_t$ at different frequency level can be calculated as

$$P_t(\omega, \lambda) = \sqrt{\left(\sum_{j=1-t}^{T-t} b_{t,j+t} \cos(\omega j)\right)^2 + \left(\sum_{j=1-t}^{T-t} b_{t,j+t} \sin(\omega j)\right)^2} \qquad (6)$$

Bloechl (2014) explained that this gain can be interpreted as a factor by which the amplitude of an oscillation with a certain frequency is decreased or increased by a filter. Taking T=50 we plotted this gain (P_t) in Figure 2 (Appendix B) for t=1, 3, 25, 26, 47, and 49. By construction, weights in HP filter are different for middle and terminal points. We can see in this figure how gain is affected by changing the weights (i.e. for middle and terminal points): for the estimations at the middle (like, 25th and 26th) we see very similar gain as these depend upon an almost equal weight structure and the gain starts to change as we move to the boundaries (like for the 1st, 3rd, 47th and 49th estimations). Thus, for the estimation at end points the high frequencies cannot be completely eliminated anymore which causes an increasing volatility in the trend component. So the trend estimates at terminal points contain part of the cyclical component and are thus distorted[6]. To quantify the distortion at the terminal data points in trend component Bloechl (2014) introduced a 'loss measure' in the form of deviation of gain function of certain estimation $\hat{g}_t$ from the one at the middle (centre) where we know distortion is negligible. This loss is what one would like to minimize so that the distortion at the terminal points is not more than that at the middle of the dataset (i.e., negligible). If $P_c(\omega, \lambda)$ denotes the gain for frequency ω and parameter λ for the centre estimation $\hat{g}_c$, where c=T/2, and $P_t(\omega, \lambda)$ is the gain for estimation $\hat{g}_t$ then the loss function is

$$l(t, \lambda) = \sum_{i=1}^{n} [P_c(\omega_i, \lambda) - P_t(\omega_i, \lambda)]^2 . \theta \qquad (7)$$

We use ω=(0, 0.1, 0.2, . . ., π)/ . Here, n is number of elements in ω and θ is the distance between the element in ω i.e. $\theta = \omega_j - \omega_{j-1} = 0.1$. Calculating the loss for t=1, 2, . . .,T gives an overview of distortions at the estimations (for the trend) on terminal points. For T=50, loss is significantly high at

[4] The ideal weighting scheme is at the centre of the data set only as shown in the Figure 1 of the Appendix B.
[5] Again, we have shown this in Figure 1 of Appendix B.
[6] Resulting in larger than should be standard deviation in trend component (and hence lower than should be standard deviation in cyclical component).

terminal points when we use HP filter, with lambda=100 (as shown in Figure 3 of Appendix B). To eliminate the EPB, the ideal weighting scheme would be the one which gives zero overall loss. We critically evaluate the weighting scheme of Bloechl (2014) and propose our weighting scheme to address the EPB in section 4 and 5 respectively.

4. Review of Earlier Proposals in the Literature to Address EPB in HP Filtering

There are various ways proposed in the literature to circumvent the end point bias problem described above and a few to escape the issue (for example Mohr (2005), Kaiser and Maravall (1999), Denis et al (2002), Bruchez (2003), and Bloechl (2014)). One way to handle this issue is to extend data from both ends (Mohr (2005)) before applying HP filter to decompose the time series of interest. There are different ways to extend data. Kaiser and Maravall (1999) and Denis et al (2002) suggested to use ARIMA (p,d,q) model for extrapolating the data at both ends. Mise et al (2005) also suggested using forecast-augmentation approach to mitigate EPB. In a recently published study, Iqbal and Hanif (2017) assessed the performance of various filtering approaches [like MHP filter, wavelet analysis (WAN) based filtering and empirical mode decomposition (EMD)[7]] to manage EPB by extrapolating the underlying time series. While using forecasting-augmentation approach, they observed that the HP filter performed worse (at end points), than other filtering techniques studied. We think this is not a proper solution of end point biasness in HP filtering as the choice of data generating process (DGP) for extending the subject series in itself could be biased simply because we do not know the true DGP of the series of interest. Bruchez (2003) criticized extending the time series to circumvent the EPB in HP filtering process. He argued that unexpected behavioural changes in the underlying series will not be captured by such extrapolations.

Bruchez (2003) proposed a new mechanism of changing the weighting scheme in the HP filter: for certain values of t, g_t term appears less often in the second part of equation (2) so the corresponding value of λ for those values of t should increase. More specifically, since first and last value appears only once, 2nd and 2nd last value appears twice and all other values appears three times in second part of equation (2), he proposed to multiply lambda by 3 for first and last values, and by 3/2 for second and second last values. We believe that Bruchez (2003) approach to handle EPB also has shortcomings including a) use of arbitrary numbers (3 and 3/2) to change the weights for the terminal points, and b) ignoring the weighting issues in other than the four terminal values.

Bloechl (2014) suggested another solution of end point problem in HP filter. He introduced a new weighting scheme for the end values of the data but not arbitrarily like Bruchez (2003). He introduced a loss function to be minimized (as discussed above). In order to resolve end points asymmetrical weighting issue of HP filter, Bloechl (2014) suggested (i) flexible scheme for number of end observations (k) to consider and (ii) flexible weights for end observations (α).

With the loss function (Equation 7) one can assess the distortion which causes the biasness at the terminal points' estimates (of trend using HP filter). Bloechl (2014) developed a scheme to reduce this distortion: higher the loss, higher the penalization (in linear manner). Thus, Bloechl (2014) suggested a flexible penalization for HP filter by taking different values of λ for different points in time. Considered a cumulative loss function:

$$L(\lambda) = \sum_{t=1}^{T} l(t, \lambda) \qquad (8)$$

[7] Huang et al (1998)

Bloechl (2014) implemented his scheme by replacing the scalar λ in equation (3) with a vector λ_t while increasing λ for k values at both ends. That is

$$\lambda_{T-2-k+j} = \lambda_{HP} + \alpha j, \text{ and } \lambda_{k-j+1} = \lambda_{HP} + \alpha j, \quad j = 1, \dots, k \tag{9}$$

With different choices of α and k for new vector λ_t in equation (9), one can obtain different values of accumulated loss function. Based upon his simulation work Bloechl (2014) has given different choices of α and k for different values of λ and for different time period. Bloechl (2014) found estimated loss with his suggested scheme (1.16872) lower than that for HP filter approach (1.76382).

There are various issues in the way Bloechl (2014) attempted to address the EBP. First of all it considers only linear penalization to minimize the loss function in equation 8 whereas Figures 1, 2 and 3 (Appendix B) clearly suggest possibility of nonlinear penalization to minimize the loss function to zero. Moreover, selecting seemingly optimal weight (α) from amongst the arbitrarily (and thus exogenously) chosen values for this weight does not ensure the loss minimization.

In the following section we propose and put to test a *'fully' modified HP filter* which not only addresses these issues with ways the EPB has been attempted to address but also the issue of fixing the smoothing parameter across countries/series and over the time period.

5. Fully Modified HP Filter

We marry the endogenous lambda approach of McDermott (1997) with loss function minimization approach of Bloechl (2014) while suggesting some intuitive changes in his weighting scheme (we critically evaluated above). We contribute by suggesting an endogenous weighting scheme along with endogenous smoothing parameter and resolve EPB issue of HP filter. We call this *fully modified* (FMHP) *filter*.

Instead of having a fixed value of smoothing parameter like in Hodrick and Prescott (1997) and (the starting point of) Bloechl (2014) we estimate the lambda endogenously. For this, we estimate $g_{t,k}(\lambda)$ (i.e., equation 3) by applying the leave-out method (as used in McDermott, 1997) with λ=1 as an initial value. For different positive values of λ, we estimate equation (4) and select λ that gives the minimum value of the objective function in equation 4. We call this λ^{MHP}. By this time we have an endogenous smoothing parameter. Here we propose how to address EPB in the estimated trend corresponding to λ^{MHP}. Rather than simply following Bloechl (2014) scheme upon MHP filtering, we propose the following (improved) scheme to minimize the cumulative loss in equation 8: (i) use linear or non linear increase of penalization (whichever minimizes the cumulative loss in equation 8) to the terminal points, (ii) fix the value of k (=20)[8] and (iii) endogenize the weights (for end observations) i.e. select α endogenously.

Now, we explain the overall process to implement our FMHP filter's overall procedure in the following:

First, estimate $g_{t,k}(\lambda)$ applying the leave-out method using equation 3 starting with an arbitrary value for λ (=1);

[8] We find k=20 as best one (in a simulation exercise) instead of flexible value of k (as used by Bloechl, 2014). This makes required number of observations in a time series to be above 40 in order to apply our FMHP filter for smoothing purposes.

Second, for different values for $\lambda>0$, we obtain different estimates of (equation 4) and λ that gives the minimum value of the objective function (4) is chosen as the initial smoothing parameter as in McDermott (1997), λ^{MHP}.

Third, using value of λ^{MHP} and weighting scheme below, we obtain optimal value of α and k.

$$\lambda_{T-2-k+j}^{MHP} = \lambda^{MHP} + \alpha j^{i}, \text{ and } \lambda_{k-j+1}^{MHP} = \lambda^{MHP} + \alpha j^{i}, \quad j = 1, \dots, k; \ i=1, 2. \tag{9a}$$

Fourth, repeat first step to re-estimate $g_{t,k}(\lambda)$ using equation 3 but with our weighting scheme and optimal value of α and k from step 3;

Fifth, using $g_{t,k}(\lambda)$ from step 4 along with different values of λ and new weighting scheme below we obtain different values of $g_{t,k}(\lambda)$ and hence different values of equation (4). And λ that gives the minimum value of the objective function (4) is chosen as the optimal smoothing parameter, λ^{FMHP}.

$$\lambda_{T-2-k+j}^{FMHP} = \lambda^{FMHP} + \alpha j^{i}, \text{ and } \lambda_{k-j+1}^{FMHP} = \lambda^{FMHP} + \alpha j^{i}, \quad j = 1, \dots, k; \ i=1, 2. \tag{9b}$$

Sixth and last, using optimal λ and new weighting scheme we estimate the $\text{trend}^{\text{FMHP}}$ and then deduce the $\text{cyclical}^{\text{FMHP}}$ component[9].

Before we go into implementation of our FMHP filtering upon artificial and observed data, we would like to present the evidence that our weighting scheme improves upon HP filter. From Figure 1a (of the Appendix B) we can observe that in case of our proposed weighting scheme, the weights at the end points are not much different from those at the middle of the data set. We can recall from Figure 1 (of the Appendix B) that in HP filter scheme the terminal point weights are significantly large compared to those at middle points. We also observe that the range (maximum minus minimum weight) for the terminal points is significantly reduced in our scheme to 0.182 (from 0.385 of HP filter) and it converges to the range at middle points (Figure 1a, Appendix B).

Moreover, our weighting scheme improves the gain function and reduces the associated (cumulative) loss compared to HP filtering approach. If we look at Figure 2b (in comparison with Figure 2) of Appendix B, we can observe that the gain function for terminal points converges to zero as we approach the higher frequency. Similarly, we can see that the cumulative loss function gets very close to zero at end points as shown in the Figure 3b (in comparison with Figure 3) of Appendix B. In Figures 2b and 3b (of the Appendix B) we have used k=20 and $\alpha = 10$. We found $\alpha = 10$ by minimizing the loss function in equation 8 with our proposed weighting scheme.

Interestingly, these observations are true when we endogenize the weight (α) and keep value of lambda fixed. We know, once we endogenize lambda our estimated cyclical component improves significantly (as shown in Choudhary et al, 2014).

In the following we show the performance of FMHP filtering based upon simulation exercise (section 5.1) and application to real world dataset (section 5.2).

[9] A MATLAB code for estimating the trend and cyclical components using our FMHP filter is available on MathWorks File Exchange website at https://www.mathworks.com/matlabcentral/fileexchange/63198-fully-modified-hp-filter-function

5.1. Simulation

Our simulation exercise has two stages: (i) generation of artificial series, and (ii) use of artificially generated series to evaluate HP, BK, CF, WAN-WED and this study's FMHP filters. Following Hodrick and Prescott (1997), we know that a typical economic time series (x_t) is composed of a trend (g_t) and a cycle (c_t) i.e. $x_t = g_t + c_t, \quad t = 1, 2, 3, \ldots, T$. By choosing suitable DGP, as discussed below, trend and cyclical components are generated separately. These two components are then combined to obtain single time series. This single time series is later decomposed using each of the above listed filters (i.e. HP, BK, CF, WAN-WED, and FMHP filters). We compare the performance of these filters in extracting the cyclical part of the series. We use the root mean squared error (RMSE) as performance criterion. Ideally it should be zero. We actually see the abilities of these five filters to estimate cyclical components at end points of data series as well as in the middle in order to assess which filter performs the best particularly in minimizing the EPB.

Following Harvey and Jaeger (1993), and Guay and St. Amant (2005) the trend and cyclical components for quarterly data can be generated as[10]

$$g_t = \text{drift} + \text{trend}_t + g_{t-1} + \varepsilon_t \quad (10a)$$

$$c_t = \theta_1 c_{t-1} + \theta_2 c_{t-2} + \delta_t \quad (10b)$$

Where $\varepsilon_t \sim \text{NIID}(0, \sigma_\varepsilon^2)$, $\delta_t \sim \text{NIID}(0, \sigma_\delta^2)$.

The data-generating process of equations 10a and 10b is chosen on the evidence that the trend of most observed macroeconomic series tends to follow a random walk with a drift, which could be either linear or nonlinear, while the cyclical series follows an AR(2) process. The DGP has general specification where trend part satisfies the unit root condition while cyclical part follows the stationary process [with $\theta_1 + \theta_2 < 1$ *and* $|\theta_2| < 1$].

We also consider the change in relative importance of each component by varying the ratio of standard deviation, $\sigma_\varepsilon/\sigma_\delta$, of the disturbances in equations 10a and 10b. In order to generate the artificial data closer to some observations (we have) from real life data we take these ratios slightly different from Choudhary et.al (2014) and Guay and St. Amant (2005). We consider the following values of the ratio $\sigma_\varepsilon/\sigma_\delta$: 10, 5, 2, 1 and 0.50.

The various combinations of values we assume for parameters (θ_1 and θ_2) and the ratio of SDs in equations 10a and 10b are given in columns (b) to (d) in Table 1 (of the Appendix A). We have 30 different 'combinations of assumptions' for generating artificial data series reported in this table. Since average length of annual data series of all countries (in this study) is about 50 years, so against each model/DGP we take 200 observations for quarterly type data set and repeat this process 1000 times. We use different time aggregation methods to convert high frequency (quarterly) artificial data into low frequency (annual) data - namely systematic (every 4th value from quarterly data series), summing (taking sum of 4 consecutive values from quarterly series) and averaging (taking average of 4 consecutive values from quarterly series).

Table 2 (Appendix A) carries the results of average RMSE of cyclical components extracted from HP, BK, CF and FMHP filters for whole data set as well as for the middle (80% length of the series) of the data and end points (10% from both ends). First of all we see if there is any difference of RMSE between middle and terminal points of the series. In Table 2 (Appendix A) we can see that RMSE for

[10] Like in our earlier study (Choudhary et al, 2014) on evaluating MHP filter of McDermott (1997).

end points is significantly higher than the RMSE at the middle of the data set for all the filtering techniques used here. Hence for both quarterly and annual data series, all these filters have upward bias at end points of the series - for example, in case of quarterly data set the RMSE of HP filter at 'terminal points' is 350% high than the RMSE at the middle. For FMHP filter this increase is 145% which is less than half (as compared to HP filter's 350%). Hence FMHP filter has smallest bias while HP and CF filters have higher EPB for quarterly and annual data respectively. In Table 2a (Appendix A) we compare RMSE of cyclical components estimated using FMHP filter and WAN-WED filtering. We find that our filtering approach performs better than any ad-hoc solution like extension of subject time series to 'sideline' the EPB. Thus, from Tables 2 and 2a (Appendix A) we can see that FMHP filter has 'overall' lowest RMSE as compared to other filtering methods for quarterly as well as annual data sets. Hence *EPB is reduced significantly by using FMHP filter*.

Columns (e) to (l) in Table 1 (Appendix A) carry the results of the performance comparison of FMHP filter with HP filter, a band pass filter (namely, CF filter[11]) and WAN-WED filtering. In this table we report the percentage of the times our FMHP filter performs better than HP, WAN-WED (reported in parentheses) and CF (reported in square brackets) filters for each of the 30 'combinations of assumptions'/models to generate artificial data. None of HP, WAN-WED and CF filter could beat our FMHP filter even in a single model single time in this power comparison study.

We now turn to real life data and compare the performance of FMHP based filtering with HP filter (being the most popular amongst economics researchers) to see if results from simulated study are robust in observed data application.

5.2. Empirical Application

We use annual and quarterly time series of three core macroeconomic variables namely real GDP, real (private) consumption and real investment. We select those 70 countries for which at least 40 annual observations[12] for each of these series is available. Quarterly national income accounts being scant, we could find quarterly time series for income, consumption and investment for 33 countries only[13]. The quarterly data is seasonally adjusted. We grouped all the countries into four income categories: high, upper middle, lower middle and lower income (as per World Bank 2015 classification). All the series are transformed into (natural) logarithms before we proceed to decompose the observed time series.

In the decomposition process we have seen above that the ideal filter, to address the EPB issue completely, is the one which minimizes the loss function to zero. In section 5.2.1 we see which of the HP and FMHP filters minimizes the loss[14].

Later in section 5.2.2 we put both the HP filter and FMHP filter to real life test to observe implications of using HP filter approach compared to FMHP filter for moments of detrended macroeconomic time series. For this purpose practice is to compare the univariate (like autoregressive

[11] Our overall power comparison analysis considers only one band pass filter and that is CF filter. We cannot workout BK filter for end point performance as it loses certain terminal observations by its construction.
[12] Minimum number of observations required to implement our weighting scheme.
[13] Annual data is from World Bank database, whereas quarterly data is taken from OECD dataset. Annual data starting (ending) point varies from 1960 to 1975 (2010 to 2014). Minimum time span analysed for annual (quarterly) data is 41(18) years.
[14] For the sake of completeness the results presented in the Tables 3a and 3b (Appendix A) also report the loss estimated on the basis of Bloechl (2014).

coefficients) and multivariate (like unconditional correlation coefficients) analytics of the cyclical component of relevant series.

5.2.1. Comparing the estimated loss functions

In Table 3a (3b) of Appendix A we present the estimated loss found while decomposing real income[15], consumption and investment[16] annual (quarterly) data series of 70 (33) countries. In columns b to d, the fixed values of lambda [100 (1600) for annual (quarterly)] are used to estimate the loss function given in equation (8).

In column b, we use fixed value of lambda and weighting scheme of Hodrick and Prescott (1997) to obtain loss for all countries. As we know when lambda is exogenously fixed, estimated loss is function of T (and fixed) lambda, the estimated loss for HP filter is independent of actual series and its underlying dynamics. The slight difference in estimated loss for different countries (as reported in column b) is only because of difference in number of observations for each of the country in the Tables 3a and 3b (Appendix A). However, when we apply our weighting scheme with fixed lambda (like in HP filter) we see the estimated loss reduces to 0.599 (0.801) compared to 1.734 (1.764) found while we use the weighting scheme of Hodrick and Prescott (1997) in annual (quarterly) real income series[17]. Hence, even if we use exogenous lambda, our scheme perform better than HP filter approach in addressing the EPB. We find further improvement in lowering the EPB if we apply our weighting scheme and use endogenous lambda as reported in columns e to f of Tables 3a and 3b. Thus FMHP filter of this study is best approach to minimize the EBP in HP type filtering.

5.2.2. Moments' Comparison

It is common practice in literature to compare the univariate and multivariate analytics of cyclical component of an underlying time series extracted using the filters to be evaluated. Here we appraise the cyclical components of real income, consumption and investment obtained using HP filter and FMHP filter based upon a) volatility (standard error), persistence (autoregressive coefficient), and unconditional correlation coefficients of consumption and investment with income following Baxter and King (1999).

For HP filter we know λ is used as 100 (1600) for annual (quarterly) time series. Based on these exogenous values of smoothing parameter we extract the cyclical components of each of the three selected macroeconomic series for all the countries we study. For FMHP filter, we first estimate the values of λ using the leave-one-out procedure (equation 4) along with new weighting scheme (equation 8 with endogenous λ) for each of the annual as well as quarterly income, consumption and investment series for all the selected countries. Smoothing parameter now could be different for different periods of time even for the same series of a country and for end/middle points of the same

[15] In case of all the filters for which we compare the estimated loss.

[16] In case of FMHP filter only. As a matter of fact when lambda is exogenously fixed, estimated loss is function of T (and fixed) lambda (see equation 8) only. Thus estimated loss for HP filter is independent of actual series and its underlying dynamics. It remains the same even if our weighting scheme is applied while using fixed lambda. Though our scheme will reduce the estimated loss but lambda is exogenous. Only when our scheme is applied to MHP filter, i.e. when lambda is endogenous, the estimated loss is function of underlying time series (and its dynamics). Thus in case of FMHP filter we have different estimated loss for each of the income, consumption and investment series.

[17] We can see Bloechl (2014) scheme (with fixed lambda), reduces the estimates loss to some extent but not as largely as our scheme (with fixed lambda) as reported in columns d of Tables 3a (3b) for annual (quarterly) real income series

series[18]. Hence, it will be mistake to assume smoothing parameter to be fixed across the countries/series/time.

In the following, we show moments' implications of using fixed smoothing parameter (as in HP filter) compared to FMHP filtering approach of estimating the endogenous smoothing parameter while minimizing EPB. We estimate the standard errors and first order autoregressive coefficients (AR1) of the cyclical components of observed series extracted using HP and MFHP filters. We also obtain the unconditional correlation coefficients of detrended series for pairs of interest (consumption-income and investment- income). We report (the AR(1)) coefficient's equality test (following Paternoster et al.,1998) and the Fisher's Z-test for correlation coefficient's equality (following Bundick, 1975) in Tables 4 and 5 (Appendix A). While comparing the individual detrended series analytics (Table 4, Appendix A) we observe that a) 'on average' the difference in AR(1) coefficients of detrended series using two methods (MFHP filter minus the HP filter) is positive across countries and series for annual data while for quarterly data this difference is almost zero, b) on average difference in the SEs of detrended series obtained by these filters (MFHP minus HP filter) is also positive across series and countries and frequency (especially for annual data) indicating less of cyclical component is left in trend when we extract cycle using FMHP filter[19], and c) the AR(1) coefficients of a cyclical part of a time series obtained from two approaches are statistically significantly different from each other across the countries and series for annual data. While comparing the unconditional correlation coefficients (Table 5, Appendix A) we observe two important things. First, for annual data set, on the average the point estimates of cross correlation coefficients between the cyclical components extracted by FMHP filter of the income-consumption and income-investment pairs are higher than those between the cyclical components extracted using the HP filter. However, the opposite is true for quarterly data correlations. Second, although the point estimate difference between pair wise correlation coefficients are small for both annual and quarterly data set, most of these differences are statistically significant. For both annual and quarterly data, there are about 60 percent countries having statistically significant pair wise correlation difference. This shows that the choice of λ and weighting scheme are also relevant for second order moments of annual series.

6. Conclusion

Despite its extensive use to extract cyclical component from a macroeconomic time series, end point bias issue of (fixed smoothing parameter based) Hodrick and Prescott (1997) filter is well documented in the relevant literature. EPB in the estimated business cycles keep economic managers in the dark about the true state of their economies.

Bloechl (2014) observed that the reason for EPB issue is non-symmetrical weighting scheme in HP filtering process for significant number of terminal observations. Bloechl (2014) suggested varying number of end observations upon which he applied different weights (than HP filter) and found some reduction in EPB.

In this study, we further the McDermott (1997) Modified HP filter of endogenous smoothing parameter by combining it with an intuitive weighting scheme to solve EPB in HP filtering. We propose to use linear or non linear increase of penalization (whichever minimizes the cumulative loss) to the terminal points while fixing the number of end point observations to penalize and endogenous 'end point observations' weights'. We call our filtering approach a fully modified HP (FMHP) filter.

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[18] That is why we cannot have one lambda for all the data points of a series like in HP filter and MHP filter.

[19] Here we show how we solve the main implication of EPB highlighted in footnote 6.

In a power comparison study, our FMHP filter outperforms a) the conventional filters (like HP, BK and CF filter), and b) ad-hoc solution of EPB like wavelet analysis based filtering with extrapolated time series. End point performance of our FMHP filter is specifically evaluated and found best amongst a set of competing filtering approaches. When we put FMHP filtering to real life test based detrending (of real income, consumption and investment time series of a large number of countries) we find that our filter significantly lowers the EPB compared to Bloechl (2014) and that it performs better in moments' analytics compared to HP filter.

With the use of better estimates of the cyclical behavior (with FMHP filtering) of their economies, economic managers will have better knowledge of the state of their economic dynamics and thus will be able to take necessary stabilization measures at right time.

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Appendix A

Table 1: Simulation Results of Performance[1] Comparison of fully modified HP filter with HP, WAN-WED, and CF fitters[2]

Model	$(\sigma_\varepsilon/\sigma_\xi)$	AR Coefficients: First ($\emptyset_1$)	AR Coefficients: Second ($\emptyset_2$)	Percent of times fully modified HP filter outperforms HP, (WAN-WED[6]), [CF[7]] filters: (Generated as) Quarterly: Linear trend	(Generated as) Quarterly: Non linear trend	Time Aggregated (Annual): Systematically: Linear trend	Systematically: Non linear trend	By Summing: Linear trend	By Summing: Non linear trend	By Averaging: Linear trend	By Averaging: Non linear trend
(a)	b[3]	c[3]	d[3]	e[4]	f[4]	g[5]	h[5]	i[5]	j[5]	k[5]	l[5]
1	10	0.9	0.01	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
2	10	1.2	-0.25	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
3	10	1.2	-0.4	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
4	10	1.2	-0.55	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
5	10	1.2	-0.75	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
6	5	0.9	0.01	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
7	5	1.2	-0.25	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
8	5	1.2	-0.4	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
9	5	1.2	-0.55	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
10	5	1.2	-0.75	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
11	2	0.9	0.01	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
12	2	1.2	-0.25	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
13	2	1.2	-0.4	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
14	2	1.2	-0.55	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
15	2	1.2	-0.75	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
16	1	0.9	0.01	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
17	1	1.2	-0.25	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
18	1	1.2	-0.4	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
19	1	1.2	-0.55	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
20	1	1.2	-0.75	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
21	0.5	0.9	0.01	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
22	0.5	1.2	-0.25	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
23	0.5	1.2	-0.4	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
24	0.5	1.2	-0.55	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
25	0.5	1.2	-0.75	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
26	10	0.8	0	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
27	5	0.8	0	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
28	2	0.8	0	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
29	1	0.8	0	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]
30	0.5	0.8	0	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]	100, (100), [100]

1: Performance criterion is the root mean square error of the artificial cyclical and estimations of those artificial cyclical series. 2: HP, WAN-WED and CF denote Hodrick-Prescott, Wavelet Analysis based filtering (with extrapolated data) and Christiano-Fitzgerald filters. 3. Columns b-d presents model's assumptions for generating the artificial data. 4. Columns e-f are the power of fully modified HP filter compared to HP, WAN-WED and CF filters for quarterly data generated by linear and non-linear models respectively. 5. Columns g to l represent the power of fully modified HP filter compared to HP, WAN-WED and CF filters for time aggregated (by summing as well as averaging) annual data generated by linear and non-linear models. 6. For WAN-WED filter, we extrapolated the underlying time series using AR(1) model. 7. For CF filter we use maximum length of cycle P1 =32 for quarterly and P1=8 for annual data; minimum length of cycle P2=6 for quarterly data and P2=2 for annual data.

Table 2: Root Mean Square Error of Cyclical component estimated by Fully Modified HP, HP, BK and CF filter

		Average RMSE of cyclical component of 30 models											
		Generated data set (full)				Generated data set (middle values 80%)				Generated data set (end points 20%)			
		FMHP	HP	BK	CF	**FMHP**	HP	BK	CF	**FMHP**	HP	BK	CF
(Generated as) Quarterly		**3.0**	13.0	39.9	57.1	**2.3**	3.8	19.9	17.3	**5.6**	17.1	NA	73.9
Time Aggregated (Annual)	Systematically	**8.7**	33.8	18.1	33.7	**3.6**	17.0	17.8	10.9	**17.4**	67.4	NA	72.2
	By Summing	**23.3**	135.1	72.6	135.9	**10.1**	68.4	71.3	43.5	**46.3**	269.4	NA	291.2
	By Averaging	**5.9**	33.8	18.2	34.0	**2.5**	17.1	17.9	10.9	**11.7**	67.3	NA	72.8

Table 2a: Root Mean Square Error of Cyclical component estimated by Fully Modified & Wavelet Analysis with extrapolation (WAN WE)

		Average RMSE of cyclical component of 30 models					
		Generated data set (full)		Generated data set (middle values)		Generated data set (end points)	
		FMHP	WAN (WE)	**FMHP**	WAN (WE)	**FMHP**	WAN (WE)
(Generated as) Quarterly		**3.0**	8.3	**2.3**	8.9	**5.6**	78.2
Time Aggregated (Annual)	Systematically	**8.7**	24.3	**3.6**	23.9	**17.4**	92
	By Summing	**23.3**	57.0	**10.1**	56.3	**46.3**	337
	By Averaging	**5.9**	14.5	**2.5**	14.4	**11.7**	76

Table 3a: Estimated Loss (Annual Data)

	Fixed Lambda (100)			Proposed weighting scheme with endogenous Lambda		
	HP Filter	HP (Bloechl scheme)	HP (our scheme)	FMHP filter	FMHP filter	FMHP filter
	Income	Income	Income	Income	Consumption	Investment
Algeria	1.735	1.260	0.626	0.545	0.532	0.560
Australia	1.735	1.260	0.626	0.523	0.560	0.539
Austria	1.733	1.267	0.545	0.513	0.491	0.494
Bangladesh	1.735	1.260	0.626	0.521	0.535	0.637
Belgium	1.733	1.267	0.545	0.506	0.507	0.468
Benin	1.735	1.260	0.626	0.537	0.525	0.540
Bolivia	1.735	1.260	0.626	0.586	0.567	0.533
Botswana	1.731	1.147	0.647	0.505	0.501	0.505
Brazil	1.735	1.260	0.626	0.561	0.565	0.536
Burkina Faso	1.732	1.272	0.623	0.508	0.482	0.484
Cameroon	1.735	1.260	0.626	0.576	0.534	0.606
Canada	1.733	1.267	0.545	0.520	0.501	0.476
Chile	1.735	1.260	0.626	0.556	0.529	0.522
Colombia	1.735	1.260	0.626	0.522	0.529	0.580
Congo, Rep	1.735	1.260	0.626	0.536	0.602	0.539
Costa Rica	1.735	1.260	0.626	0.546	0.546	0.552
Cuba	1.733	1.256	0.534	0.482	0.482	0.507
Cyprus	1.731	1.147	0.647	0.471	0.391	0.391
Denmark	1.733	1.267	0.545	0.492	0.522	0.470
Dominican Republic	1.735	1.260	0.626	0.541	0.527	0.542
Ecuador	1.735	1.260	0.626	0.561	0.541	0.537
Egypt	1.733	1.225	0.604	0.470	0.423	0.397
El Salvador	1.732	1.272	0.623	0.513	0.506	0.503
Finland	1.733	1.267	0.545	0.517	0.481	0.540
France	1.733	1.267	0.545	0.505	0.506	0.489
Gabon	1.735	1.260	0.626	0.522	0.529	0.522
Germany	1.733	1.267	0.545	0.477	0.472	0.504
Greece	1.733	1.267	0.545	0.550	0.531	0.480
Guatemala	1.735	1.260	0.626	0.584	0.582	0.526
Honduras	1.735	1.260	0.626	0.523	0.521	0.560
Hong Kong	1.733	1.227	0.546	0.484	0.481	0.403
India	1.735	1.260	0.626	0.526	0.534	0.524
Indonesia	1.735	1.260	0.626	0.550	0.530	0.522
Iran	1.735	1.260	0.626	0.565	0.593	0.566
Ireland	1.733	1.267	0.545	0.480	0.488	0.512
Italy	1.733	1.267	0.545	0.477	0.476	0.474
Japan	1.733	1.256	0.534	0.487	0.483	0.481
Kenya	1.733	1.262	0.627	0.503	0.606	0.494
Lesotho	1.733	1.262	0.627	0.512	0.532	0.498
Luxembourg	1.733	1.267	0.545	0.476	0.510	0.509
Madagascar	1.735	1.260	0.626	0.521	0.546	0.548
Malaysia	1.735	1.260	0.626	0.529	0.620	0.560
Malta	1.733	1.225	0.604	0.474	0.375	0.366
Mauritania	1.735	1.260	0.626	0.528	0.521	0.560
Mexico	1.735	1.260	0.626	0.562	0.526	0.523
Morocco	1.734	1.278	0.623	0.522	0.510	0.541
Netherlands	1.733	1.267	0.545	0.481	0.486	0.505
New Zealand	1.733	1.267	0.545	0.484	0.479	0.472
Norway	1.735	1.260	0.626	0.568	0.526	0.551
Pakistan	1.735	1.260	0.623	0.568	0.526	0.551
P.N. Guinea	1.733	1.256	0.534	0.489	0.487	0.457
Peru	1.735	1.260	0.626	0.571	0.561	0.590
Philippines	1.735	1.260	0.626	0.583	0.572	0.542
Portugal	1.733	1.267	0.545	0.504	0.486	0.472
Puerto Rico	1.733	1.227	0.546	0.471	0.471	0.475
Rwanda	1.735	1.260	0.626	0.531	0.531	0.539
Senegal	1.735	1.260	0.626	0.522	0.527	0.563
Singapore	1.731	1.147	0.647	0.470	0.417	0.469
South Africa	1.735	1.260	0.626	0.582	0.569	0.566
South Korea	1.735	1.260	0.626	0.559	0.544	0.531
Spain	1.733	1.267	0.545	0.536	0.545	0.494
Sudan	1.735	1.260	0.626	0.525	0.521	0.557
Sweden	1.733	1.267	0.545	0.509	0.487	0.473
Thailand	1.735	1.260	0.626	0.552	0.521	0.529
Togo	1.735	1.260	0.626	0.524	0.678	0.523
Trinidad & Tobago	1.735	1.260	0.626	0.581	0.575	0.595
Tunisia	1.734	1.278	0.623	0.508	0.497	0.502
UK	1.733	1.267	0.545	0.516	0.492	0.510
Uruguay	1.732	1.272	0.623	0.484	0.481	0.528
USA	1.733	1.256	0.534	0.495	0.482	0.445
Venezuela	1.733	1.262	0.627	0.500	0.497	0.496
Average	**1.734**	**1.256**	**0.599**	**0.522**	**0.518**	**0.514**

Table 3b: Estimated Loss (Quarterly Data)

	Fixed Lambda (100)			Proposed weighting scheme with endogenous Lambda		
	HP Filter	HP (Bloechl scheme)	HP (our scheme)	FMHP filter	FMHP filter	FMHP filter
	Income	Income	Income	Income	Consumption	Investment
Australia	1.762	1.618	0.818	0.829	0.881	0.815
Austria	1.761	1.259	0.786	0.616	0.741	0.806
Belgium	1.766	1.495	0.797	0.649	0.686	0.616
Brazil	1.761	1.259	0.786	0.798	0.698	0.800
Canada	1.760	1.608	0.818	0.659	0.669	0.683
Costa Rica	1.769	1.682	0.814	0.657	0.720	0.911
Czech Rep	1.761	1.259	0.786	0.582	0.641	0.652
Denmark	1.766	1.495	0.797	0.547	0.608	0.571
Estonia	1.766	1.495	0.797	0.589	0.594	0.642
Finland	1.769	1.682	0.818	0.710	0.729	0.684
France	1.762	1.617	0.820	0.610	0.668	0.573
Germany	1.769	1.682	0.814	0.747	0.798	0.637
Greece	1.766	1.495	0.797	0.685	0.694	0.696
Hungary	1.766	1.495	0.797	0.623	0.644	0.739
India	1.757	1.125	0.770	0.663	0.750	0.651
Ireland	1.757	1.125	0.770	0.622	0.582	0.633
Italy	1.761	1.259	0.786	0.661	0.709	0.677
Korea	1.762	1.617	0.818	0.812	0.840	0.784
Latvia	1.766	1.495	0.797	0.560	0.526	0.629
Lithuania	1.766	1.495	0.797	0.627	0.580	0.642
Mexico	1.768	1.629	0.810	0.741	0.741	0.819
Netherlands	1.761	1.259	0.786	0.553	0.626	0.609
New Zealand	1.770	1.629	0.827	0.617	0.644	0.668
Norway	1.761	1.620	0.821	0.751	0.646	0.733
Portugal	1.766	1.495	0.797	0.648	0.706	0.671
Slovak Rep	1.757	1.125	0.770	0.622	0.626	0.677
Slovenia	1.766	1.495	0.797	0.574	0.762	0.625
South Africa	1.762	1.618	0.818	0.745	0.699	0.709
Spain	1.766	1.495	0.797	0.590	0.637	0.609
Sweden	1.768	1.629	0.810	0.651	0.779	0.631
Switzerland	1.762	1.617	0.820	0.670	0.655	0.646
UK	1.766	1.495	0.797	0.580	0.625	0.637
US	1.766	1.495	0.797	0.544	0.566	0.539
Average	**1.764**	**1.480**	**0.801**	**0.653**	**0.681**	**0.679**

Table 4: Net AR(1) Coefficients and Standard Errors

Country Group→	High Income			Upper Middle Income			Lower Middle Income			Lower Income		
Series[1]	*Y*	*C*	*I*	*Y*	*C*	*I*	*Y*	*C*	*I*	*Y*	*C*	*I*
Annual Data												
Number of Countries	30			16			19			5		
Average of $(\beta^f - \beta^h)$[2]	0.23	0.20	0.24	0.22	0.27	0.24	0.23	0.26	0.22	0.27	0.28	0.30
Average of $(\sigma^f - \sigma^h)$[3]	0.02	0.02	0.05	0.03	0.03	0.07	0.02	0.03	0.06	0.02	0.03	0.08
Countries *not* passing *Z*-test at 10% for H_0 : $\beta^f - \beta^h = 0$	11	11	9	7	9	6	8	10	6	3	3	2
Quarterly Data[4]												
Number of Countries	28			4			1			0		
Average of $(\beta^f - \beta^h)$	-0.02	-0.02	-0.03	0.00	0.00	0.02	-0.01	0.07	-0.00	-	-	-
Average of $(\sigma^f - \sigma^h)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00			
Countries *not* passing *Z*-test at 10% for H_0: $\beta^f - \beta^h = 0$[4]	0	0	0	0	0	0	0	0	0	-	-	-

Notes: 1. Y, C and I denote detrended income, consumption and investment series. 2. The average of the net difference in the AR(1) coefficients where superscript f and h denote fully modified HP filter and HP filter respectively. 3. The average of the net difference of the standard deviation of detrended series, where σ^f and σ^h are standard deviation of cyclical component estimated by fully modified HP filter and HP filter respectively.4. AR(1) coefficient equality tests.

Table 5: Net Unconditional Correlations

Country Group→	High Income		Upper Middle Income		Lower Middle Income		Lower Income	
Pairs[1]	Y-C	Y-I	Y-C	Y-I	Y-C	Y-I	Y-C	Y-I
Annual Data								
Number of Countries	30		16		19		5	
Average of $(\rho_i^f - \rho^h)$[2]	0.04	0.03	0.02	0.02	0.10	0.06	-0.02	0.00
Countries *not* passing *Z*-test at 10% for H^0: $\rho_i^f - \rho^h = 0$	21	12	9	8	11	9	3	2
Quarterly Data								
Number of Countries	28		4		1		0	
Average of $(\rho_i^f - \rho^h)$	-0.04	-0.03	0.01	0.00	0.04	-0.02	-	-
Countries *not* passing *Z*-test at 10% for H^0 $\rho_i^f - \rho^h = 0$[3]	18	16	2	3	0	0	-	-

Notes: 1. Y-C and Y-I denote unconditional correlations of individually detrended income-consumption and income-investment pairs. 2. The average of net of the correlation coefficients $(\rho_i^f - \rho^h)$ where the correlation coefficient: ρ^f and ρ^h are obtained from wavelet and modified HP filter separately. 3. Correlation equality tests.

Appendix B

Figure 1: Weighting Vectors of HP Filter with Lambda 100

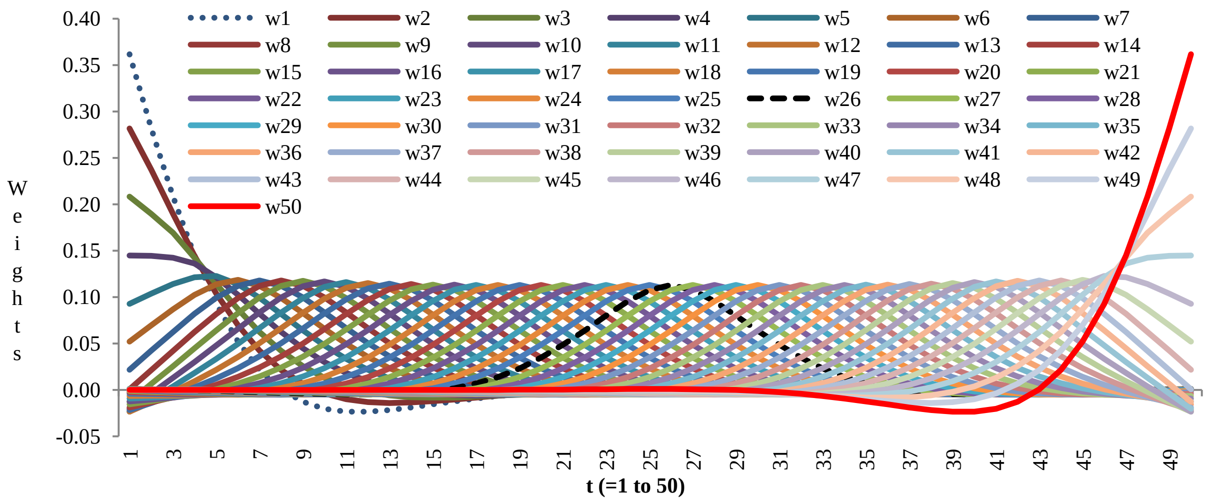


Figure 1a: Weighting Vectors of our Weighting Scheme Filter with Lambda 100

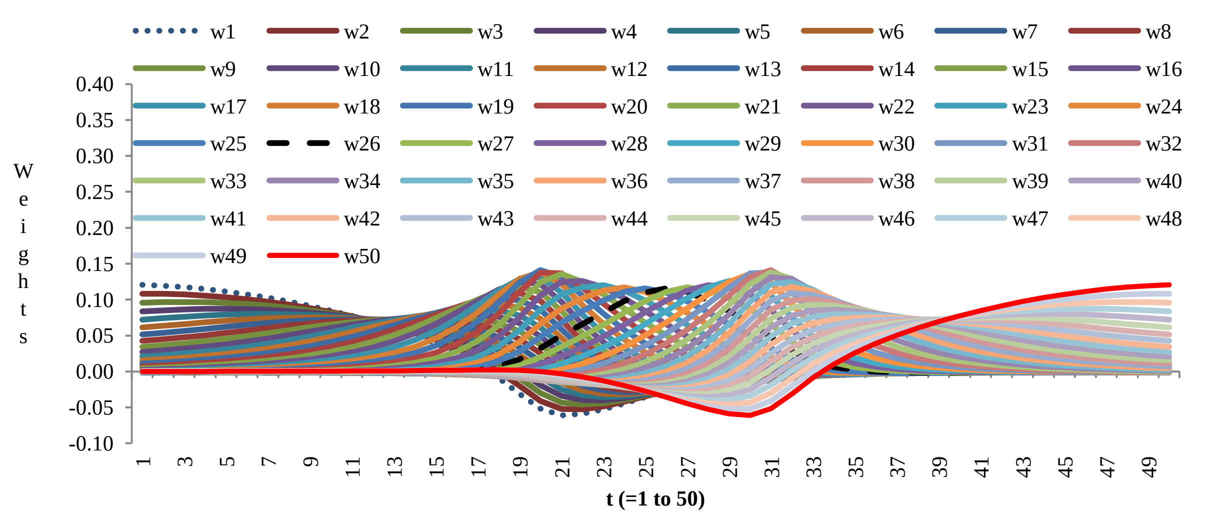
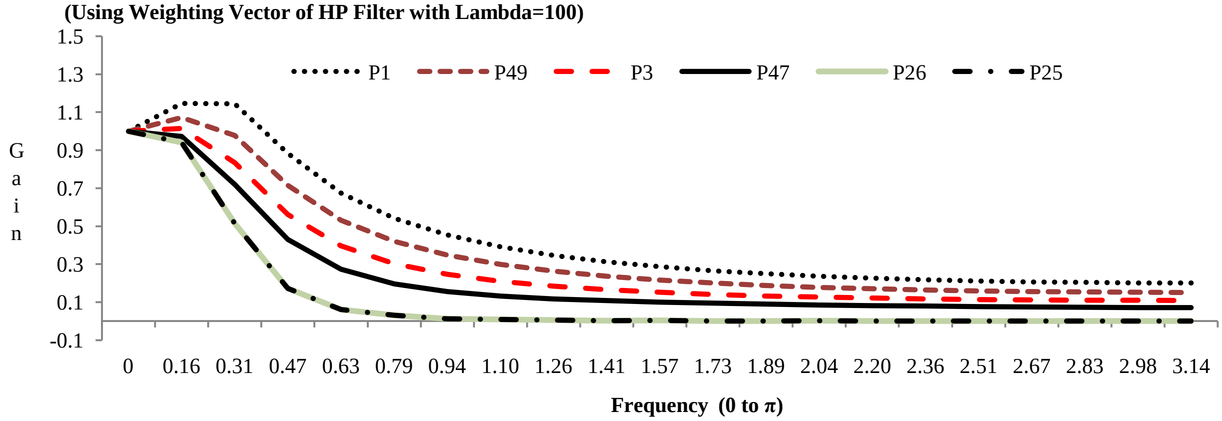


Figure 2: Gain Function for Terminal and Middle Observations
(Using Weighting Vector of HP Filter with Lambda=100)



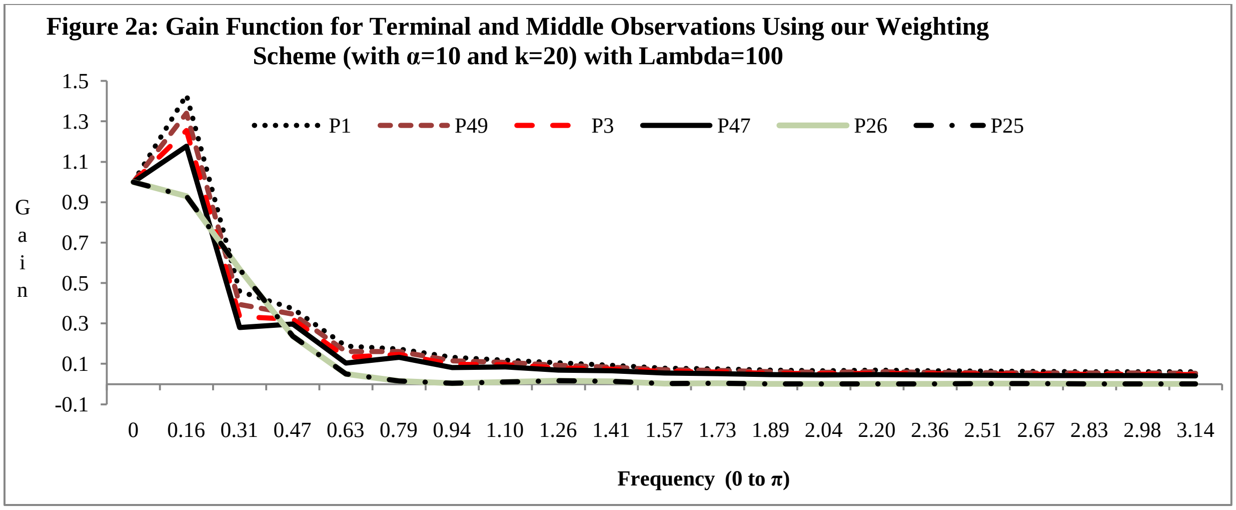


Figure 2a: Gain Function for Terminal and Middle Observations Using our Weighting Scheme (with α=10 and k=20) with Lambda=100
P1
P49
P3
P47
P26
P25
Gain
1.5
1.3
1.1
0.9
0.7
0.5
0.3
0.1
-0.1
0 0.16 0.31 0.47 0.63 0.79 0.94 1.10 1.26 1.41 1.57 1.73 1.89 2.04 2.20 2.36 2.51 2.67 2.83 2.98 3.14
Frequency (0 to π)

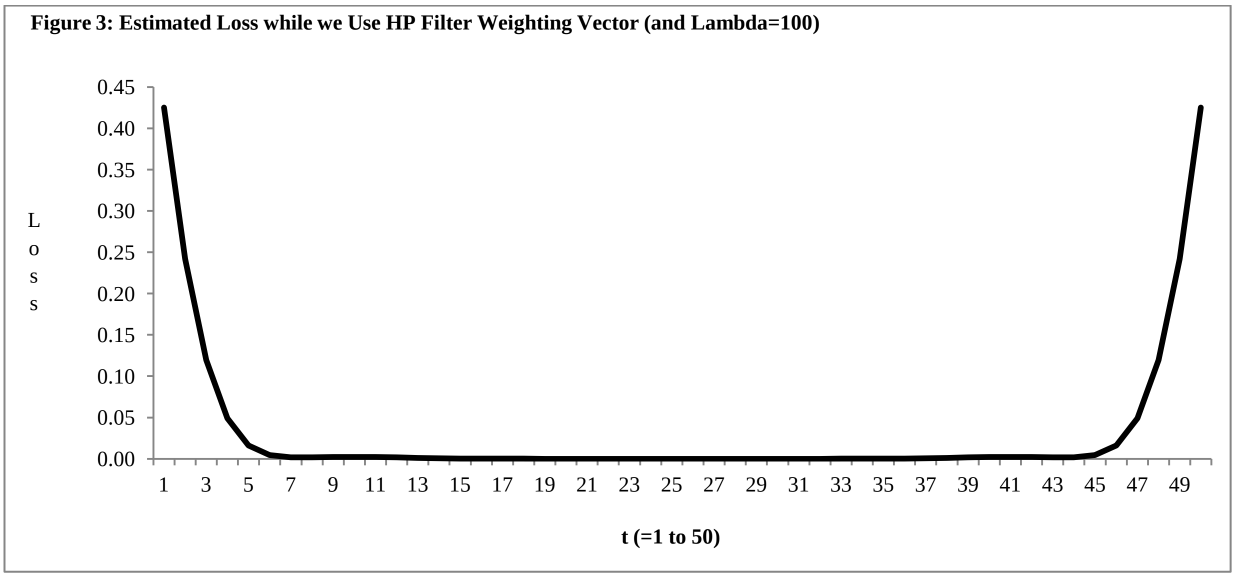


Figure 3: Estimated Loss while we Use HP Filter Weighting Vector (and Lambda=100)
Loss
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
1 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 35 37 39 41 43 45 47 49
t (=1 to 50)

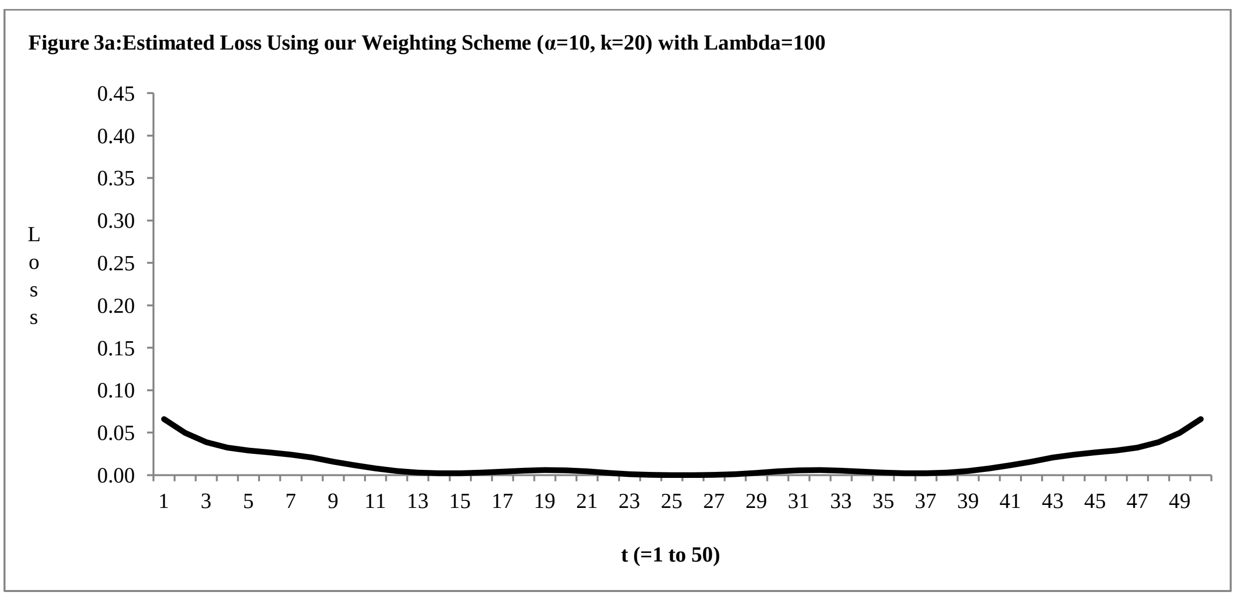


Figure 3a:Estimated Loss Using our Weighting Scheme (α=10, k=20) with Lambda=100
Loss
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
1 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 35 37 39 41 43 45 47 49
t (=1 to 50)

Appendix C

The solution of the minimization problem (in equation 2 of the main body of this paper) is explained below in more detail [than given in Danthine and Girardian (1989)].

$\min_{g_t} [\sum_{t=1}^{T}(x_t - g_t)^2 + \lambda \sum_{t=2}^{T-1}[(\mathrm{g_{t+1}} - \mathrm{g_t}) - (\mathrm{g_t} - \mathrm{g_{t-1}})]^2]$ (C1)

$= \min_{g_t} [\sum_{t=1}^{T}(c_t)^2 + \lambda \sum_{t=2}^{T-1}(\mathrm{g_{t+1}} - 2\mathrm{g_t} + \mathrm{g_{t-1}})^2]$

$= \min_{g_t} [\sum_{t=1}^{T}(c_t)^2 + \lambda \sum_{t=2}^{T-1}(Kg_t)^2]$

$= \min_{g_t}[\ c_t^{'}c_t + \lambda(Kg_t)^{'}(Kg_t)]$ (C2)

Where

$$Kg_t = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_T \end{bmatrix}$$

We want to show that solution of equation (C2) is reached when

$x_t = [I + \lambda K^{'}K]g_t = [I + \lambda A]g_t = Bg_t$

We note that B is symmetric, since

$B^{'} = [I + \lambda K^{'}K]^{'} = I^{'} + \lambda(K)^{'}(K^{'})^{'} = I + \lambda K^{'}K = B$

Also B is positive definite, since for any vector z:

$$z^{'}Bz = z^{'}(I - \lambda K^{'}K)z = z^{'}z - \lambda(Kz)^{'}(Kz) = \sum_{i=1}^{n} z_i^2 + \lambda \sum_{i=1}^{n-s}(\Delta^{s}z_i)^2 \ \geq 0$$

Since λ is positive, both summation terms are non-negative.

Since B is positive definite, it is non-singular, hence B^{-1} exist and also positive definite.

Now from equation (C2)

$\mathrm{F} + \lambda S = (\mathrm{x_t} - \mathrm{g_t})^{'}(\mathrm{x_t} - \mathrm{g_t}) + \lambda(\mathrm{Kg_t})^{'}(\mathrm{Kg_t})$

$= \mathrm{x_t^{'}x_t} - \mathrm{x_t^{'}g_t} - \mathrm{g_t^{'}x_t} + \mathrm{g_t^{'}g_t} + \lambda\mathrm{g_t^{'}K^{'}Kg_t}$

$= \ g_t^{'}(I + \lambda K^{'}K)g_t + x_t^{'}x_t - x_t^{'}g_t - g_t^{'}x_t$

$= \ g_t^{'}Bg_t + x_t^{'}x_t - x_t^{'}g_t - g_t^{'}x_t$

$$= g_t' B B^{-1} B g_t + x_t' x_t - x_t' B^{-1} B g_t - g_t' B^{-1} B x_t$$

$$= (g_t' B - x_t') B^{-1} (B g_t - x_t) + x_t' x_t - x_t' B^{-1} x_t$$

$$= (B g_t - x_t)' \; B^{-1} (B g_t - x_t) + x_t' x_t - x_t' B^{-1} x_t$$

Since 2nd and 3rd terms are constant, only first term have unknown g_t. Hence $F + \lambda S$ is smallest when the first term is smallest. Since B^{-1} is positive definite, the first term is non-negative, its smallest value is zero: $B g_t - x_t = 0, \Rightarrow g_t = B^{-1} x_t$.