SBP Working Paper Series

No. 87 March, 2017

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STATE BANK OF PAKISTAN

SBP Working Paper Series

Editor: Muhammad Nadim Hanif

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Karachi 74000. Pakistan.

Email: wps@sbp.org.pk

ISSN 1997-3802 (Print)
ISSN 1997-3810 (Online)

Published by State Bank of Pakistan, Karachi, Pakistan.
Printed at the SBP BSC (Bank) – Printing Press, Karachi, Pakistan.

Performance Comparison of Modified HP Filter, Wavelet Analysis and Empirical Mode Decomposition for Smoothing Macroeconomic Time Series

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Abstract

We compare performance of modified HP filter, wavelet analysis and empirical mode decomposition. Our simulation study results suggest that modified HP filter performs better for an overall time series. However, in the middle (of time series) wavelet analysis performs best. Wavelet analysis based filtering has highest 'end points bias (EPB)'. However, it performs better when we extrapolate the subject time series to lower the EPB. Study based on observed data of real income, investment and consumption shows that the autoregressive properties and multivariate analytics of cyclical components depend upon filtering technique.

Keywords: Business Cycle, Smoothing Macro Time Series, Modified HP Filter, Wavelet Analysis, End Point Bias in HP Filter, Simulation, Cross Country Study.

JEL Classification: E32, C18

Acknowledgements

The authors are thankful to two anonymous reviewers for their valuable suggestions which helped imporve the study.

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Non-technical Summary

In the macroeconomic research studies it is important to estimate the business cycle in as best manner as possible. Correctly estimated business cycles help economists to better analyze fluctuations caused by different types of economic activities. Moreover, properly estimated business cycles give economists a quality benchmark to examine the validity or otherwise of the theoretical models.

We compare performance of the recently developed filters – modified HP filter, wavelet analysis and empirical mode decomposition – for smoothing macroeconomic time series using artificial as well as observed data. Our simulation study results suggest that modified HP filter performs better compared to other two filters for an overall time series. This is due to higher 'end points bias (EPB)' when we use wavelet analysis based filtering. Wavelet analysis filtering performs better only in the 'middle' of a time series. In order to lower the EPB if we extrapolate the subject time series (at both ends), wavelet analysis performs better for the subject time series. Our study based on observed annual as well as quarterly data of real income, investment and (private) consumption for a large number of countries shows that the autoregressive properties and multivariate analytics of cyclical components depend upon filtering techniques.

We find wavelet analysis based filtering performing better if we extrapolate the subject observed time series using ARIMA(p,d,q) model.

1. Introduction

In the macroeconomic research studies it is important to estimate the business cycle in as good manner as possible. There are at least two reasons for this. First, correctly estimated business cycle helps economists to better analyze fluctuations caused by different types of economic activities. Second, properly estimated business cycle provides a quality benchmark to examine the validity or otherwise of the theoretical models (Canova, 1998).

Most of the macroeconomic time series are composed of a trend, seasonal variation, cyclical fluctuation and irregular component. A large number of statistical methods have been proposed to decompose a time series into its components. Once a series is seasonally adjusted, the most popular method amongst the business cycle researchers, to estimate the cyclical component, is the Hodrick-Prescott (1997) filter (HP filter). Hodrick-Prescott (1997) suggests fixing the value of the smoothing parameter, λ, at 1600 (100) for quarterly (annual) series.

Although HP filter is a popular choice for detrending a series, the practice of fixing the value of λ across the series and countries remain a controversial issue (Choudhary et al (2014)). Fixing the smoothing parameter across the countries/series amounts to ignoring the country and variable specific behavior of underlying economic agents which actually should determine λ. To resolve this issue, Choudhary et al (2014) assessed the performance of modified HP filter (MHP) of McDermott (1997). On the basis of artificially generated data they compared modified HP filter with the HP filter, Baxter and King (1999), and Christiano and Fitzgerald (2003) filters. It was found that irrespective of simulation model assumptions, data frequency, and aggregation method (for annual data), the modified HP filter performs better and that the choice of detrending method matters in actual/observed data analytics.

Kaiser and Maravall (1999) and Ekinci et al. (2013), however, raised two serious questions about HP type filtering techniques: the end points biased (EPB) and the amount of noise in the cyclical component. Since MHP filter is a modified form of HP filter we suspect MHP filter may have said drawbacks. There exist various approaches in the literature to address these two questions including a) extending the dataset before applying filtering to see if EPB fades away, b) opting for decomposition method which also extracts noise (in addition to cyclical component) like the wavelet analysis (WAN) so that noise part is not distributed in extracted cycle and trend, and c) choosing altogether another filter like empirical mode decomposition (EMD)[1] by Huang et al (1998). In this study we test these three approaches to see which one addresses these questions in a better way.

We contribute to the relevant literature in following ways. We design and conduct a 'simulation study' as well as 'observed data' analysis to evaluate the performance of MHP filter against WAN based filtering and EMD. We investigate the biasness at terminal points by using artificially generated data. We see if there is any reduction in EPB by extending data (with, say, AR1 model). Using observed economic time series of three core macroeconomic variables, namely the real income, investment and (private) consumption, for 125 (33) countries having annual (quarterly) data, we evaluate the performance of top two filters in our simulations study - the MHP filter and WAN based filter. For this purpose we examine the sensitivity of standard deviations (SDs) and degree of persistence in the cyclical components to the choices of detrending methods and compare the impact of the choice of smoothing techniques on the unconditional correlation between the real income-real investment and real income-real consumption pairs.

[1] It can be any: Baxter and King (1999) or Christiano and Fitzgerald (2003).

Here we highlight some of our results. First, with 1000 artificially generated series, we observe that MHP filter performs better on overall data set especially at end points while WAN based filtering outperforms in the middle of data set. Second, with artificially generated series, we find that EPB is highest for WAN and lowest for MHP filter approach. By extending data set from both ends, WAN outperforms the other two methods in extracting cyclical component and EMD perform worst. Third, for real data, we find that there is statistical significant difference in AR(1) coefficients of cyclical component extracted from MHP and WAN based filtering. Fourth, we observe that pair wise correlations of cyclical components extracted by WAN based filtering are significantly greater than their counterparts based upon cyclical components extracted by MHP filter. Finally, we can say that the cyclical component obtained from WAN based filtering is free of noise.

Remainder of this study is organised as follows: In Section 2, we provide the procedures to decompose a time series using (i) modified HP filter, (ii) wavelet analysis, and (iii) empirical mode decomposition (from the literature). In Section 3, we brief the end point bias and pick of one of the various solutions – extrapolating the subject time series – suggested in the literature to address this bias. In Section 4, we draw a simulation study to evaluate the three filtering techniques (with extrapolating the artificially generated time series) to address EPB. In Section 5, we put these filtering techniques to test and decide which one performs better with real life data (of a large number of countries). In Section 6 we conclude.

2. Review of Literature

2.1 Modified HP Filter

According to Hodrick and Prescott (1997), a seasonally adjusted time series is divided into permanent (trend) and transitory (cyclical) parts so that

$$y_t = g_t + c_t, \qquad t = 1, 2, 3, \ldots, T \tag{1}$$

Where y_t, g_t and c_t denote a time series and its trend and cyclical parts respectively. HP filter estimates a cyclical series (c_t) by minimizing the sum of square of difference between series (y_t) and its trend part (g_t) subject to the constraint that the squared sum of dynamic differences of the trend is not too large. The optimization problem is given below

$$\min\left[\sum_{t=1}^{T}(y_t - g_t)^2\right] \quad \text{(goodness of fit)}$$

Subject to

$$\sum_{t=1}^{T}(\Delta^2 g_t)^2 = \sum_{t=1}^{T}[(g_{t+2} - g_{t+1}) - (g_{t+1} - g_t)]^2 = \mathrm{v} \quad \text{(degree of smoothness)}$$

where Δ^2 is the second-order difference (of the smooth part) and ν is a constant. The standard method to solve this problem assumes that $\nu = 0$ so that using the Lagrange multiplier we get

$$min[\sum_{t=1}^{T}(x_t - g_t)^2 + \lambda \sum_{t=2}^{T-1}[(g_{t+1} - g_t) - (g_t - g_{t-1})]^2] \tag{2}$$

In this optimization problem, there is a trade-off between the goodness of fit and the degree of smoothness that depends on the value of λ. The solution to the minimization problem in (2) for g_t is

$$\hat{g}_t = [I + \lambda A]^{-1} y_t \tag{3}$$

Where $A = K'K$ where $K = \{k_{ij}\}$ is a (T-2) ×T matrix with elements as given below

$$k_{ij} = \begin{cases} 1 & \text{if } j = i \text{ or } j = i + 2, \\ -2 & \text{if } j = i + 1, \\ 0 & \text{otherwise} \end{cases}$$

Hodrick and Prescott (1997) apply this procedure on quarterly US GDP data by fixing the value of λ at 1600 to estimate cyclical and trend component.

It has now been a convention for smoothing quarterly and annually macroeconomic series across economies and across series with $\lambda = 1600$ and 100 respectively.

Modified HP filter of McDermott (1997) relaxes this (assumption of) fixed values of λ as explained in Choudhary et.al (2014). The idea of this procedure is to apply HP filter method (in equation 3) by excluding a single data point at a time and select a λ which gives best fit of the data point left out. The emphasis therefore is on selecting an optimal value of λ with reference to the subject time series.

2.2 Wavelet Analysis Based Decomposition

Wavelets are mathematical expansions that transform data from the time domain into different layers of frequency levels. The technique has the advantage of being localized both in time and in the frequency domain, and enables the researcher to observe and analyze data at different scales.

Conventionally economists consider only two scales in a time series: the short run and the long run. There are actually more time scales in between the short run and the long run horizon of a time series (Dalkir, 2004). Through decomposition, based upon wavelet analysis, we can analyze a time series into different frequency zones, very high frequency (noise part) and very low frequency (smooth part) and we can obtain cyclical part free of noise portion. Multiresolution wavelet analysis is useful tool to decompose an economic time series into trend, cycle, and noise (Yogo, 2008). A time series y_t can be decomposed as

$$y_t = S_t^J + \sum_{j=1}^{J} D_t^j \tag{4}$$

Where S_t^J is a cycle with periodicity greater than 2^{J+1} and D_t^j denotes cycles with periodicity between 2^j and 2^{j+1}. We take J = 4 and 2 for quarterly and annual data series respectively (Yogo (2008) and Crowley (2010)). If the sampling frequency is quarterly, S_t^J is the trend component (with periodicity greater than 32). D_t^2, D_t^3 and D_t^4 are the business-cycle components with periodicity of 4-8, 8-16 and 16-32 quarters respectively and D_t^1 is a high frequency noise with periodicity less than 4 quarters. Hence for quarterly data series

$y_t = S_t^4 + \sum_{j=2}^{4} D_t^j + D_t^1$= trend component+ business cycle+ noise

2.2.1 Basics of Wavelet

A wavelet basis consists of a father wavelet that represents the smooth baseline trend and a mother wavelet that is dilated and shifted to construct different levels of details. The mother wavelet is denoted by $\psi(t)$ and must satisfy two conditions

$\int \psi(t)dt = 0$, and

$$\int |\psi|^2(t)dt = 1$$

The mother wavelet can be dilated and translated to measure the change in a function at a particular frequency and at a particular point in time.

$$\psi_{k,s}(t) = \frac{1}{\sqrt{s}}\psi(\frac{t-k}{s}) \tag{5}$$

Where k and s are the time location and scale parameters (or frequency ranges) respectively. To insure the unity norm of mother wavelet ($\psi_{k,s}(t)$), we divide it by $\sqrt{s}$ in expression (5). There are two types of wavelet transform, continuous wavelet transform (CWT) and discrete wavelet transform (DWT). The CWT can be obtained by projecting the original series y_t on mother wavelet $\psi_{k,s}(t)$.

$$W(k,s) = \int_{-\infty}^{+\infty} y_t\,\psi_{k,s}(t)dt \tag{6}$$

CWT is computationally complex and contains a high amount of unnecessary information (Benhmad (2013)). However its discrete variant (DWT) is more convenient as it needs only limited number of translated and dilated versions of mother wavelet to decompose the given series (Benhmad(2013) and Gencay et.al.(2002)). The DWT of a time series with T observation is calculated only at scales 2^j and the largest number of scales equal to the integer $J = [\log_2(T)]$

The DWT is based on two discrete wavelet filters which are called mother wavelet $h_l = h_0, h_1, \dots\dots, h_{L-1}$ and the father wavelet $g_l = g_0, g_1, \dots\dots, g_{L-1}$. The mother wavelet is a high pass (wavelet) filter while father wavelet is a low pass (scaling) filter. The mother wavelet filter h_l follows two conditions $\sum_{l=0}^{L-1} h_l = 0$ and $\sum_{l=0}^{L-1} h_l^2 = 1$. The coefficients of mother wavelet are determined by quadrature mirror relationship with father wavelet.

$$h_l = (-1)^{l+1} g_{L-1-l} \text{ for } l = 1,2,\dots\dots, L-1$$

The basic properties of scale filter are $\sum_{l=0}^{L-1} g_l = \sqrt{2}$ and $\sum_{l=0}^{L-1} g_l^2 = 1$ and $\sum_{l=0}^{L-1} g_l\, g_{l+2n} = 0$ for all n>0 (Benhmad (2013)). The scaling filters satisfy the orthonormality property as they have unit energy and are orthogonal to even shift. Thus wavelet decomposition can separate high frequency component from its low frequency component by applying low-pass and high-pass filters to a given series. The mother wavelet represents the details or high frequency components, and father wavelet captures the smooth or low frequency parts. The father wavelet is longest time-scale (2^J) component of the series (trend).

2.2.2 Maximal Overlap Discrete Wavelet Transform (MODWT)

Since DWT has some limitations, as it requires the sample size to be an integer multiple of 2^j i.e. T=2, 4, 8 or 16 ……. The number of wavelet and scaling coefficients decreases by a factor of 2 for each increasing level of the transform. These deficiencies can be overcome if the down sampling in the DWT can be avoided. This can be achieved by using the maximal overlap discrete wavelet transform (Walden, 2001). MODWT need rescaling the defining filters to conserve energy.

$$\hat{h}_{j,l} = {h_l}/{2^{j/2}} \text{ and } \hat{g}_{j,l} = {g_l}/{2^{j/2}}$$

Let $W_1(t)$ and $V_1(t)$ be the high frequency (wavelet) and low frequency (scaling) coefficients at the first scale (j=1) decomposition of a given time series y_t then

$$W_1(t) = B_1\, y_t \quad \text{And } V_1(t) = A_1\, y_t$$

And for j=2, $W_2(t) = B_2 V_1(t)$ and $V_2(t) = A_2 V_1(t)$

For stage 1 (i.e. j=1) and L=4 (for simplicity) we have

$$
B1=\begin{bmatrix}
\hat{h}_{1,0} & 0 & 0 & \cdots & 0 & 0 & \hat{h}_{1,3} & \hat{h}_{1,2} & \hat{h}_{1,1} \\
\hat{h}_{1,1} & \hat{h}_{1,0} & 0 & \cdots & 0 & 0 & 0 & \hat{h}_{1,3} & \hat{h}_{1,2} \\
\hat{h}_{1,2} & \hat{h}_{1,1} & \hat{h}_{1,0} & \cdots & 0 & 0 & 0 & 0 & \hat{h}_{1,3} \\
\hat{h}_{1,3} & \hat{h}_{1,2} & \hat{h}_{1,1} & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & \hat{h}_{1,3} & \hat{h}_{1,2} & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \hat{h}_{1,2} & \hat{h}_{1,1} & \hat{h}_{1,0} & 0 & 0 \\
0 & 0 & 0 & \cdots & \hat{h}_{1,3} & \hat{h}_{1,2} & \hat{h}_{1,1} & \hat{h}_{1,0} & 0 \\
0 & 0 & 0 & \cdots & 0 & \hat{h}_{1,3} & \hat{h}_{1,2} & \hat{h}_{1,1} & \hat{h}_{1,0}
\end{bmatrix}
$$

And for stage 2 (i.e. j=2) we have

$$
B2=\begin{bmatrix}
\hat{h}_{2,0} & 0 & 0 & \cdots & 0 & \hat{h}_{2,2} & 0 & \hat{h}_{2,1} & 0 \\
0 & \hat{h}_{2,0} & 0 & \cdots & \hat{h}_{2,3} & 0 & \hat{h}_{2,2} & 0 & \hat{h}_{2,1} \\
\hat{h}_{2,1} & 0 & \hat{h}_{2,0} & \cdots & 0 & \hat{h}_{2,3} & 0 & \hat{h}_{2,2} & 0 \\
0 & \hat{h}_{2,1} & 0 & \cdots & 0 & 0 & \hat{h}_{2,3} & 0 & \hat{h}_{2,2} \\
\hat{h}_{2,2} & 0 & \hat{h}_{2,1} & \cdots & 0 & 0 & 0 & \hat{h}_{2,3} & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \hat{h}_{2,1} & 0 & \hat{h}_{2,0} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \hat{h}_{2,1} & 0 & \hat{h}_{2,0} & 0 \\
0 & 0 & 0 & \cdots & \hat{h}_{2,2} & 0 & \hat{h}_{2,1} & 0 & \hat{h}_{2,0}
\end{bmatrix}
$$

Similarly we can obtain matrix A by replacing $\hat{h}_{j,l}$ with $\hat{g}_{j,l}$ in matrix B.

By using pyramid algorithm we can obtain frequency by frequency decomposition of original series. At second stage we further decompose low frequency component $V_1(t)$ into high frequency component $W_2(t)$ and low frequency component $V_2(t)$. At this stage decomposition of y_t looks like $W = [W_1, W_2, V_2]$. By applying pyramid algorithm again and again up to scale J we finally obtain.

$W = [W_1, W_2, W_3, \dots W_J, V_J]$.

Now we can construct original time series from wavelet and scaling coefficients W and V by applying multi-resolution analysis (MRA). We can use MRA by applying inverse MODWT on W_j and V_J (where j=1,2,3…,J), according to pyramid algorithm. The wavelet representation can be expressed as

$$y_t = B_1^t W_1 + B_1^t B_2^t W_2 + \cdots + B_1^t B_2^t \dots B_J^t W_J + A_1^t A_2^t \dots A_J^t V_J \tag{7}$$

$$= D_1(t) + D_2(t) + \cdots + D_J(t) + S_J(t) = D_1(t) + \sum_{j=2}^{J} D_j(t) + S_J(t)$$

=noise+ cycle+ trend

Where B^t is a transpose of matrix B.

A lot of wavelet families have been introduced. In this study, we will use Daubechies (1992)'s D4 wavelet which is orthogonal symmetric wavelet filter. D4 scaling function is given below

$g_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}$, $g_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}$, $g_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}$, and $g_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}$ and wavelet function is

$h_0 = -g_3$, $h_1 = g_2$, $h_2 = -g_1$ and $h_3 = g_0$

2.3 Empirical Mode Decomposition(EMD)

Empirical Mode Decomposition (EMD) is another useful technique for the analysis of fluctuation in macroeconomic time series. This methodology was first explained by Huang et.al (1998). EMD provides a better insight into structure of time series, which is useful for capturing the changing volatility of the business cycle (Kozic and Sever, 2014). EMD represents a decomposition technique by which any time series could be decomposed into a finite set of mono-component functions, called Intrinsic Mode Functions (IMF). The mono-component property means that the functions do not resemble a pattern of riding waves (Huang et.al, 1998). An IMF should satisfy the following conditions

i) The IMF series must have zero mean.
ii) The difference between the number of its extrema (maxima and minima) and the number of zero line crossing should be one.

By satisfying the above conditions, an IMF is approximately symmetric about zero and all its maxima are positive and all minima are negatives.

The step wise algorithm of EMD is given by Kozic and Sever (2014) as below:

i) Find the local minima and local maxima of the given time series y_t.
ii) Draw envelops around local maxima and local minima with cubic splines ($s_{max}(t)$ and $s_{min}(t)$).
iii) Calculate the mean value of these envelops i.e. $x_1(t) = \frac{[s_{max}(t)+s_{min}(t)]}{2}$.
iv) Extract IMF by subtracting mean $\mathrm{x_1(t)}$ from the time series $\mathrm{y_t}$. In ideal case, the first extraction should be an IMF. If it is not, we repeat the above steps by taking this extraction as the original time series until an extraction finally becomes an IMF.
v) Subtract IMF from the time series y_t i.e. $r_1(t) = y_t - IMF_1(t)$
vi) Repeat the whole above process by using $r_1(t)$ as the new time series, until the residual $r_n(t)$ is either monotonic (a trend) or constant i.e. $y_t = \sum_{i=1}^{n} IMF_i\,(t) + r_n(t)$

The final residual $\mathrm{r_n(t)}$ is either monotonic or constant. If it is monotonic, then it represents the trend of the time series and if it is constant, then the trend is represented by $\mathrm{IMF_n}$.

3. End Points Bias

In order to carry out a trend-cycle decomposition of a series at a given date, most of the filtering techniques (like HP, MHP, Band pass filters etc) require information about the behavior of the series at earlier as well as at future dates. This poses difficulties at the start and end of the sample, where only one sided observations are available (by definition), with the result that use of any filtering approach often leads to the so called end-point bias[2] (see Auria et.al, 2010, and Baxter and King, 1999). Also from time scale decomposition techniques like wavelet filter, the trend and cyclical components are obtained by weighted average of components of original time series. By definition starting values of trend components are weighted average of starting as well as end values of the series and if there is big difference between start and end value of time series then starting values of these components are biased upward and similarly end values of trend component biased downwards. One way to overcome this end point problem is to extend data from both ends (Mohr (2005)). There are different ways to extend data. We will use AR1 model for extrapolation[3]. We are not saying that

[2] For details on EPB see our forthcoming research study (SBP Working Paper).
[3] One can use ARIMA (p,d,q) model for this purpose as suggested by Kaiser and Maravall (1999) and Denis et al (2002).

this is the best solution of end point biasness as the choice of DGP for extending the subject series in itself is biased simply because we do not know the actual DGP of the subject series. However, we have shown that the EPB reduces with extrapolating the subject time series[4].

4. Simulation Exercise

There are two parts of this exercise. First we generate artificial data series and secondly evaluate trend and cyclical parts of these series by using the three filtering techniques – MHP filter, WAN based filter and EMD[5].

Assuming that a seasonally adjusted economic time series can be partitioned into a trend g_t and a cycle c_t so that $y_t = g_t + c_t, \quad t = 1, 2, 3, . . . , T$. By choosing suitable data generating process (DGP), our artificial series contain these two components and by combining them we obtain single time series.

As in Choudhary et al (2014); following Harvey and Jaeger (1993) and Guay and Amant (2005) , the trend and cyclical components for quarterly data can be generated as

$$g_t = \text{drift} + g_{t-1} + \varepsilon_t \quad (8a)$$

$$c_t = \theta_1 c_{t-1} + \theta_2 c_{t-2} + \delta_t \quad (8b)$$

Where $\varepsilon_t \sim \mathrm{NIID}(0, \sigma_\varepsilon^2)$, $\delta_t \sim \mathrm{NIID}(0, \sigma_\delta^2)$.

The DGP has general specification where trend part satisfies the unit root condition while cyclical part follows the stationary process with $\theta_1 + \theta_2 < 1$ and $|\theta_2| < 1$.

We also consider the change of relative importance of each component by varying the ratios of standard deviation, $\sigma_\varepsilon/\sigma_\delta$, of the disturbances in equations 8a and 8b. As in Choudhary et.al (2014) and Guay and St. Amant(2005), we consider the following values of the ratios; 10, 5, 1, 0.5, and 0.01.

Various combinations of parameters (θ_1 and θ_2) and the ratio of SDs in equation 8a and 8b are given in columns (b) to (d) in table 2. We have 30 different sets of assumptions for generating an artificial series reported in the rows of table 2. Since average length of annual data series of all countries in this study is about 50 years, so against each model/DGP we take 200 observations for quarterly data set and repeat this process 10,000 times. We use different time aggregation methods to convert high frequency artificial (quarterly) data into low frequency (annual) data; namely systematic (every 4th value from quarterly data series), summing (taking sum of 4 consecutive values from quarterly series) and averaging (taking average of 4 consecutive values from quarterly series)[6].

After generating (artificial) trend and cyclical components we combine them to get overall time series. We use the three methods under study to separate this time series back into a trend and a cycle. We compare the performance of these filters in extracting the cyclical part of the series. We use the root mean squared error (RMSE) as performance criterion. Ideally it should be zero. We actually see the abilities of these filters to estimate cyclical components at end points of data series as well as on the

[4] The best solution would be to minimize the EPB (to zero) without resorting to extending the subject time series. For this purpose we have '*fully*' modified the HP filtering in (our) another study (*forthcoming SBP Working Paper*).

[5] Most of this section is based upon Choudhary et al (2014)

[6] These three methods are important to use in this study because these are the common types of aggregation observed in real life macroeconomic time series: summing in case of GDP, systematic in case of price indices, and averaging in case of exchange rate, for example.

middle portion of the data in order to assess EPB. Objective is to reach the filter which has lowest EPB.

Table 1 carries the results of average RMSE of cyclical component extracted from MHP filter, WAN filter and EMD approach both for whole data set and for end points (5% from both ends). We also present RMSE for extended data set in table 1. From Table 1, it is clear that RMSE for end points is greater than RMSE for whole data set. Hence for both quarterly and annual data series, all these filters have upward bias at end points of the series; MHP filter has small upward bias while wavelet and EMD approaches have higher EPB.

However, by extending/extrapolating data from both ends (to overcome this end point problem), there is an improvement in WAN results. RMSE from WAN based filtering for quarterly as well as annual data is the smallest one.

Columns (e) to (l) in Table 2 carry the results of the comparison of WAN based filter with MHP filter and EMD. The results in these columns are the percentage of the times WAN based filter performs better compared to the other two filters for given set of assumption for equation 8a and 8b.

Our findings suggest that irrespective of simulation model, data frequency and aggregation method, the modified HP filter dominated for overall data set as well as at the terminal points. When we extrapolated the artificially generated data series (at both ends), to overcome EPB, the WAN based filtering showed outstanding performance - out of 30 DGPs, WAN based filtering is superior than MHP filter in 26 models. MHP filtering has no significant change in EPB when we extended the time series (as it already uses series optimal smoothing parameter). EMD turned out to be least performing in the settings of this study. We now turn to real life data and compare the performance of WAN based filtering with MHP filter to see if results from simulated study are robust.

5. Empirical Application

Evaluation of real business cycle of an economic time series involves matching moments of the model with the relevant detrended economic series. The general practice is to compare the autoregressive coefficients and unconditional correlations of cyclical component of relevant series. Therefore, in this section we evaluate cyclical components of a time series from MHP filter and WAN based filter and will see how autoregressive coefficients (in univariate analysis) and unconditional correlation coefficients (in multivariate analysis) are affected by the choice of the filtering technique. Our simulation study show that there was significant improvement in WAN based filtering when we extended data.

We use annual and quarterly (seasonally adjusted) series of real GDP, real (private) consumption and real investment. We take annual data series for 125 countries from World Bank data base. Quarterly data series for 33 countries is obtained from Organization for Economic Cooperation and Development (OECD). The data is transformed into logarithms. For annual frequency the starting points varies from 1960 to 1995 whereas the end point is 2014. The shortest data span we take is 20 years. There are relatively fewer countries having quarterly data set; most series end at 2014 and shortest data span is 18 years.

We divided all the countries into four income groups; high income; upper middle income; lower middle income and lower income (according to World Bank classification).

Like in Choudhary et al (2014), we obtain the first order autoregressive coefficients and standard errors of the cyclical components of observed series using the two filtering techniques found better in above simulation study (i.e. WAN filter and MHP filter). We also obtain the unconditional correlations of detrended series for pairs of interest (income-consumption and income-investment). We report (the AR(1)) coefficient's equality test (following Paternoster et al, 1998) and the Fisher's Z-test for correlation coefficient's equality (following Bundick, 1975)[7]. In Table 3, the results of individual series are reported while Table 4 presents differences in unconditional-correlations.

While comparing the individual detrended series analytics (Table 3) we observe that a) 'on average' the difference in AR(1) coefficients of detrended series using two methods (WAN filter minus the MHP filter) is positive across countries, series and frequencies, b) on average difference in the SDs of detrended series obtained by these filters (Wavelet minus MHP filter) is negative across series and countries and frequency (especially for annual data) [8], and c) the AR(1) coefficients of a cyclical part of a time series obtained from two approaches are statistically significantly different from each other across the countries, series and frequency [9] especially for quarterly data.

While comparing the unconditional correlation coefficients (Table 4) we would like to highlight two main findings. First, for quarterly data set, the point estimates of cross correlation coefficients between the cyclical components extracted by WAN filter of the income-consumption and income-investment pairs are higher than those between the cyclical components extracted using the MHP filter. For annual data also, same is true except for income-investment pairs for middle income counties. Second, although the point estimate difference between pair wise correlation coefficients are small for both annual and quarterly data set, some of these differences are statistically significant. This evidence is stronger for quarterly data set where for almost all the courtiers this difference is statistically significant. For annual data, there are about one-third countries in each income group having statistically significant pair wise correlation difference (in favour of WAN filtering).

6. Concluding Remarks

Popular detrending techniques usually extract 'noise loaded' cyclical component which also has end point bias. It seriously distorts the interpretation of the cycle obtained for business cycle research. The wavelet analysis (WAN) based filtering provides time-scale (frequency) analysis of a time series. We can decompose a time series into different frequency levels including high frequency (noise), middle frequency (cycles) and low frequency (trend) parts. However, wavelet analysis based filtering produces higher end point bias compared to other recently developed filters like Modified HP filter (see Choudhary et al, 2014). In an attempt to compare the performance of three recent filtering approaches – WAN based filter, MHP filter and Empirical Mode Decomposition – we find that MHP filter performs better for overall data set (especially at end points) while wavelet filter outperforms on middle part of data set in reproducing artificial cyclical series in a variety of data generating schemes. When we extend the subject times series using AR1 model we observe that WAN based filtering performs best amongst the filters studied in this paper. We also put these findings from simulation study to test in real life data of three core macroeconomic variables (income, investment and

[7] See Yu and Dunn (1982) for details.

[8] This we think is due to the fact that the WAN filter extracts cyclical component free of noise part (the high frequency component). WAN treats noise separately as against the other filtering techniques as shown in Figures A1 to A3 of Appendix (using real GDP of Australia). MATLAB codes to estimate trend, cycle and noise components using WAN filter are available. There are separate codes for i) without and ii) with extending (based on AR1 model) the subject time series.

[9] Especially for quarterly data sets.

consumption) of a large number of countries (133 in case of annual time series and 33 in quarterly time series). We find that the AR(1) coefficients of cyclical components obtained by MHP filter are biased downward. SDs of cyclical components obtained by MHP filter are on higher side (may be because it is loaded with the noise part). In terms of levels of persistence of detrended macroeconomic series, the choice of detrending techniques matter. Moreover, pair wise correlation coefficients obtained by different filtering methods are also significantly different from each other (especially for quarterly data) and that such pair wise correlations are higher in cases where WAN based filtering is used to extract the cyclical components.

Thus, we can rely upon modified HP filter to extract cycle component if we have to use available dataset only (in case we think extrapolation will create another type of bias). When we are willing to extrapolate the available dataset to reduce end point bias, wavelet analysis based filtering is better option.

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Annexure

Table 1. Root Mean Square Error of Cyclical Component Estimated by Wavelet, Modified HP, and EMD Filter

		Average RMSE of Cyclical Component of All 30 Models								
		Generated Data Set (Full)			Generated Data Set (End Points)			Extended Data		
		Wavelet	MHP	EMD	Wavelet	MHP	EMD	Wavelet	MHP	EMD
(Generated as) quarterly		27.21	10.64	156.82	78.18	10.70	413.14	8.29	11.07	19.37
Time aggregated (annual)	Systematically	57.51	33.07	162.51	92.00	34.00	262.88	24.33	36.85	130.89
	By summing	163.51	123.64	669.04	337.00	126.00	1223.46	57.04	136.19	550.32
	By averaging	37.74	32.65	188.29	76.00	33.00	344.51	14.48	35.32	99.73

Table 2. Simulation Results of Performance Comparison of wavelet filter with modified HP and EMD filter

Model	$(\sigma_\varepsilon/\sigma_\xi)$	AR coefficient		Percent of times when Wavelet filter outperforms modified HP, (EMD) filter							
		First ($\emptyset_1$)	Second ($\emptyset_2$)	(Generated as) quarterly		Time aggregated (annual)					
						Systematically		By summing		By averaging	
				Generated data set	Extended data set	Generated data set	Extended data set	Generated data set	Extended data set	Generated data set	Extended data set
(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)	(k)	(l)
1	10	0.9	0.01	20,(40)	90,(80)	10,(80)	90,(100)	50,(80)	100,(100)	50,(70)	100,(100)
2	10	1.2	-0.25	40,(60)	90,(80)	10,(40)	100,(90)	40,(90)	100,(100)	50,(90)	100,(100)
3	10	1.2	-0.4	20,(70)	90,(80)	20,(70)	100,(80)	50,(70)	100,(100)	40,(80)	100,(100)
4	10	1.2	-0.55	10,(40)	100,(90)	10,(80)	100,(100)	30,(80)	100,(100)	40,(80)	100,(100)
5	10	1.2	-0.75	20,(60)	100,(50)	30,(70)	100,(100)	60,(70)	100,(100)	60,(80)	100,(100)
6	5	0.9	0.01	20,(70)	90,(80)	20,(60)	100,(100)	40,(90)	100,(100)	30,(70)	100,(100)
7	5	1.2	-0.25	20,(50)	100,(90)	40,(80)	100,(100)	80,(100)	100,(100)	40,(90)	100,(100)
8	5	1.2	-0.4	0,(40)	100,(80)	30,(50)	100,(100)	90,(90)	100,(100)	50,(90)	90,(100)
9	5	1.2	-0.55	20,(50)	100,(80)	40,(70)	100,(100)	50,(100)	100,(100)	40,(80)	100,(100)
10	5	1.2	-0.75	0,(50)	100,(80)	20,(70)	100,(80)	50,(90)	100,(100)	70,(90)	100,(100)
11	1	0.9	0.01	10,(50)	100,(80)	50,(40)	100,(90)	40,(80)	100,(100)	50,(80)	100,(100)
12	1	1.2	-0.25	0,(30)	100,(60)	60,(80)	90,(90)	70,(100)	100,(100)	20,(40)	100,(100)
13	1	1.2	-0.4	0,(40)	100,(50)	10,(60)	100,(100)	60,(70)	100,(100)	60,(90)	100,(100)
14	1	1.2	-0.55	20,(70)	100,(80)	60,(70)	100,(80)	40,(80)	100,(100)	50,(90)	100,(90)
15	1	1.2	-0.75	30,(50)	100,(50)	30,(50)	100,(80)	40,(90)	100,(100)	60,(90)	100,(100)
16	0.5	0.9	0.01	0,(50)	100,(60)	40,(60)	100,(70)	50,(90)	100,(100)	40,(70)	100,(100)
17	0.5	1.2	-0.25	20,(30)	60,(90)	60,(70)	100,(70)	50,(70)	100,(100)	40,(80)	100,(100)
18	0.5	1.2	-0.4	50,(60)	100,(100)	40,(30)	100,(60)	20,(60)	100,(100)	60,(70)	100,(100)
19	0.5	1.2	-0.55	10,(30)	100,(90)	40,(40)	100,(60)	50,(70)	100,(80)	50,(70)	100,(90)
20	0.5	1.2	-0.75	10,(10)	100(100)	30,(40)	100,(50)	50,(70)	100,(90)	70,(80)	100,(90)
21	0.01	0.9	0.01	0,(70)	0,(100)	20,(100)	40,(100)	30,(80)	40,(60)	30,(70)	20,(90)
22	0.01	1.2	-0.25	0,(100)	0,(100)	20,(90)	20,(90)	0,(80)	0,(80)	10,(60)	10,(90)
23	0.01	1.2	-0.4	0,(90)	0,(100)	50,(60)	100,(100)	40,(60)	100,(90)	50,(70)	90,(100)
24	0.01	1.2	-0.55	0,(80)	10,(100)	40,(60)	100,(100)	80,(80)	100,(100)	80,(60)	100,(100)
25	0.01	1.2	-0.75	20,(80)	50,(100)	90,(50)	100,(80)	50,(50)	100,(90)	60,(40)	100,(100)
26	10	0.8	0	10,(60)	100,(60)	10,(90)	100,(90)	30,(60)	100,(100)	10,(70)	100,(100)
27	5	0.8	0	20,(70)	90,(70)	20,(70)	90,(90)	30,(70)	100,(100)	30,(90)	100,(100)
28	1	0.8	0	10,(30)	100,(0)	30,(80)	100,(70)	30,(60)	100,(100)	40,(70)	100,(100)
29	0.5	0.8	0	40,(40)	100,(70)	50,(60)	100,(100)	70,(80)	100,(100)	70,(80)	100,(100)
30	0.01	0.8	0	0,(80)	0,(100)	40,(70)	100,(100)	80,(80)	90,(80)	50,(90)	80,(70)

Table 3: Net AR(1) Coefficients and SDs

Country Group→	High Income			Upper Middle Income			Lower Middle Income			Lower Income		
Series[1]	*Y*	*C*	*I*	*Y*	*C*	*I*	*Y*	*C*	*I*	*Y*	*C*	*I*
Annual Data												
Number of Countries		48			31			33			13	
Average of $(\beta^{w}-\beta^{m})^{2}$	0.09	0.09	0.11	0.07	0.11	0.14	0.12	0.16	0.14	0.10	0.13	0.14
Average of $(\sigma^{w}-\sigma^{m})^{3}$	-0.01	-0.02	-0.05	-0.02	-0.02	-0.07	-0.01	-0.03	-0.09	-0.01	-0.02	-0.07
Countries *not* passing *Z*-test at 10% for H_0 : $\beta^{w}-\beta^{m}=0$	1	2	1	0	2	3	3	4	2	1	2	2
Quarterly Data[4]												
Number of Countries		28			4			1			-	
Average of $(\beta^{w}-\beta^{m})$	0.12	0.19	0.18	0.14	0.23	0.18	0.17	0.23	0.21	-	-	-
Average of $(\sigma^{w}-\sigma^{m})$	0.00	0.00	-0.01	0.00	0.00	-0.01	0.00	0.00	0.00			
Countries *not* passing *Z*-test at 10% for H_0: $\beta^{w}-\beta^{m}=0^{4}$	7	19	16	1	4	2	0	1	1	-	-	-

Notes: 1. Y, C and I denote detrended income, consumption and investment series. 2. The average of the net difference in the AR(1) coefficients where superscript w and m denote wavelet and modified HP filter. 3. The average of the net difference of the standard deviation of detrended series. Where σ^{w} and σ^{m} are standard deviation of cyclical component estimated by wavelet and modified HP filter respectively.4. AR(1) coefficient equality tests.

Table 4: Net Unconditional Correlations

Country Group→	High Income		Upper Middle Income		Lower Middle Income		Lower Income	
Pairs[1]	Y-C	Y-I	Y-C	Y-I	Y-C	Y-I	Y-C	Y-I
Annual Data								
Number of Countries	48		31		33		13	
Average of $(\rho_i^{w}-\rho^{m})^{2}$	0.03	0.01	0.03	-0.02	0.03	-0.01	0.03	0.06
Countries *not* passing *Z*-test at 10% for H^0: $\rho_i^{w}-\rho^{m}=0$	14	17	13	14	17	14	5	4
Quarterly Data								
Number of Countries	28		4		1		-	
Average of $(\rho_i^{w}-\rho^{m})$	0.04	0.04	0.02	0.02	0.12	0.06	-	-
Countries *not* passing *Z*-test at 10% for H^0 $\rho_i^{w}-\rho^{m}=0^{3}$	20	25	4	4	1	1	-	-

Notes: 1. Y-C and Y-I denote unconditional correlations of individually detrended income-consumption and income-investment pairs. 2. The average of net of the correlation coefficients $(\rho_i^{w}-\rho^{m})$ where the correlation coefficient: ρ^{w} and ρ^{m} are obtained from wavelet and modified HP filter separately. 3. Correlation equality tests.

Figure A1: Trend Component of QGDP of Australia by Wavelet Analysis

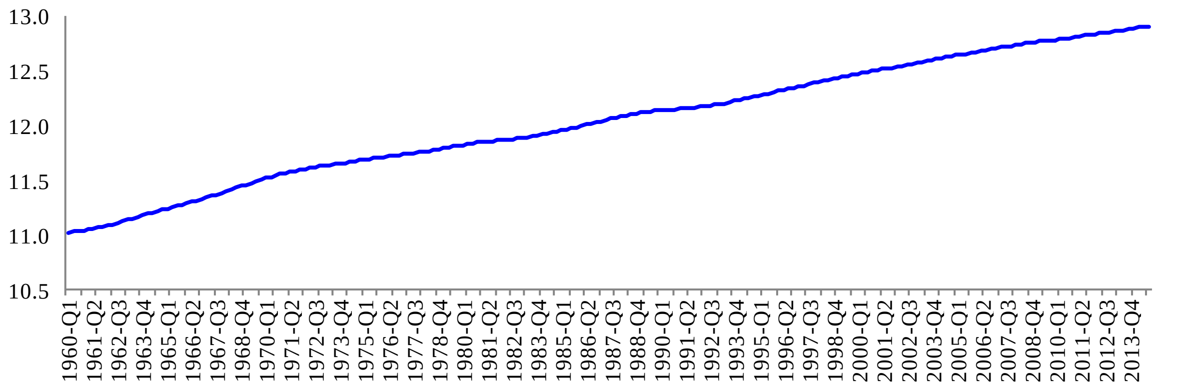

Figure A2: Cyclical Component of QGDP of Australia by Wavelet Analysis

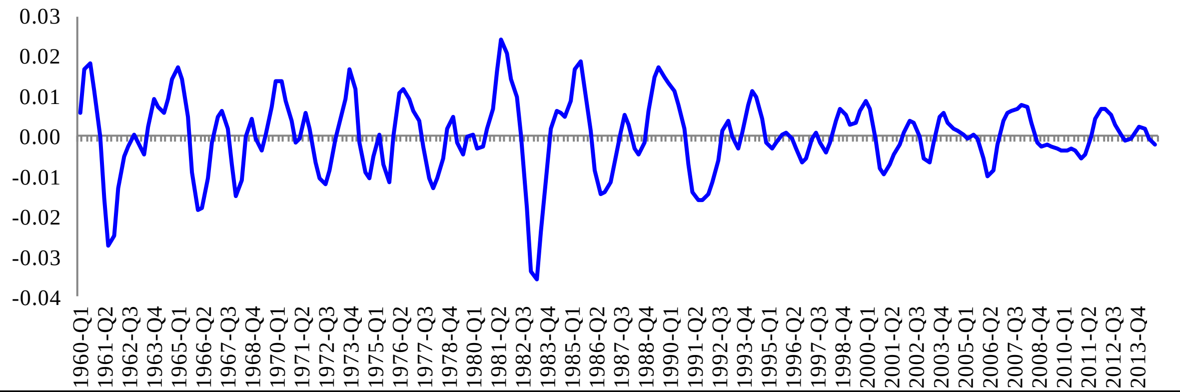

Figure A3: Noise Part of QGDP of Australia by Wavelet Analysis